Department of Mathematics
University of California, Berkeley

Mathematics 252
Representation Theory

Vera Serganova, Fall 2005

My **office hours** are 10:00-11:30 on Wednesdays and Fridays, in 709 Evans Hall. I can be reached by telephone at (64)2-2150 and electronic mail at `serganov@math`. You are welcome to ask questions by email. Homework assignments and course notes can be found on my web page **http:// math.berkeley.edu/ serganov**. First homework assignment is due on Friday, September 9

There is no required text for this course. Good references are **Fulton, Harris, Representation Theory**,**Serre, Linear Representations of Finite Groups**, **Curtis, Reiner, Representation Theory of Finite Groups and Associative Algebras**,**Gabriel, Roiter, Representations of Finite-dimensional Algebras**. I will try to post course notes regularly on the web.

To understand this course you need basic knowledge of Algebra and a good knowledge of Linear Algebra. In other words you have to know basic facts about groups and rings and you should feel very comfortable when working with linear operators.

Each Friday I will give you a problem assignment (2-4 problems) on the material of the week lectures. The homework will be collected the next Friday.

The **grade** will be computed according to the following proportions: 50% for your homework and 50% for the take home final. But if you solve all problems in your final (there will be hard ones in it) you get A for the course.

Course outline

- Representations of groups. Definitions and examples
- Schur's Lemma. Complete reducibility in case of zero characteristic
- Characters and orthogonality relation
- First examples: abelian groups, dihedral group D_n, S_3, S_4, A_5 e.t.c.
- Induced representation. Frobenius reciprocity. Mackey's criterion
- Representations of associative rings. Density theorem. Semi-simple rings, Wedderburn's theorem. Decomposition of a group algebra
- Representations of non-semisimple rings. Blocks. Injective and projective modules
- Representations of symmetric groups, Young diagrams and Frobenius formula
- Representations of general linear group, Weyl duality and Schur's polynomials (if time permits)
- Complex representations of linear groups over finite fields, Hecke algebra
- Compact groups and their representations. Peter-Weyl theorem
- Real representations and representations over subfields of C. Schur indices
- Artin's and Brauer's theorems
- Representations of quivers. Definition and examples
- Gabriel's theorem
- Representations over fields of nonzero characteristic (if time permits)

REPRESENTATION THEORY. LECTURE NOTES

VERA SERGANOVA

1. Some problems involving representation theory

Hungry knights. There are n hungry knights at a round table. Each of them has a plate with certain amount of food. Instead of eating every minute each knight takes one half of his neighbors servings. They start at 10 in the evening. What can you tell about food distribution in the morning?

Solution. Denote by x_i the amount of food on the plate of the i-th knight. The distribution of food at the table can be described by a vector $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Every minute a certain linear operator Φ is applied to a distribution x. Thus, we have to find $\lim \Phi^m$ as m approaches infinity. To find the limit we need to diagonalize Φ, and the easiest way to do this is to write

$$\Phi = \frac{T + T^{-1}}{2},$$

where T is the rotation operator:

$$T(x_1, \dots, x_n) = (x_n, x_1, \dots, x_{n-1}).$$

It is easy to see that the eigenvalues of T are the n-th roots of 1. Hence the eigenvalues of Φ are $\frac{\varepsilon^k + \varepsilon^{-k}}{2}$, where ε is a primitive root of 1, $k = 1, \dots, n$. The set of eigenvalues of Φ is

$$\left\{ \cos \frac{2\pi k}{n} \mid k = 1, \dots, n \right\}.$$

Let us chose a new basis $\{v_1, \dots, v_n\}$ in $\mathbb{C}^n$ such that

$$\Phi v_k = \cos \frac{2\pi k}{n} v_k.$$

For example, we can put $v_k = \left(\varepsilon^{-k}, \varepsilon^{-2k}, \dots, \varepsilon^{-nk}\right)$.

If n is odd all eigenvalues of Φ except 1 have the absolute value less than 1. Therefore if $x = a_1 v_1 + \dots + a_n v_n$, then

$$\lim_{m \to \infty} \Phi^m x = \lim_{m \to \infty} \sum_{k=1}^{n} a_k \left(\cos \frac{2\pi k}{n} \right)^m v_k = a_n v_n.$$

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But $v_n = (1, \dots, 1)$. Therefore eventually all knights will have the same amount of food equal to the average $\frac{x_1+\dots+x_n}{n}$.

In case when n is even the situation is different, since there are two eigenvalues with absolute value 1, they are 1 and -1. Hence as $m \to \infty$,

$$\Phi^m x \to (-1)^m a_{n/2} v_{n/2} + a_n v_n.$$

Recall that $v_k = (-1, 1, -1, \dots, 1)$. Thus, eventually food alternates between even and odd knights, the amount on each plate is approximately $\frac{a_n \pm a_{n/2}}{2}$, where

$$a_n = \frac{x_1 + \dots + x_n}{n},\ a_{n/2} = \frac{x_1 - x_2 + \dots - x_n}{n}.$$

Slightly modifying this problem we will have more fun.

Breakfast at Mars. It is well known that marsians have four arms, a standard family has 6 persons and a breakfast table has a form of a cube with each person occupying a face on a cube. Do the analog of round table problem for the family of marsians.

Supper at Venus. They have five arms there, 12 persons in a family and sit on the faces of a dodecahedron (a regular polyhedron whose faces are pentagons).

Tomography problem. You have a solid in 3-dimensional space of unknown shape. You can measure the area of every plane cross-section which passes through the origin. Can you determine the shape of the solid? The answer is yes, if the solid in question is convex and centrally symmetric with respect to the origin.

In all four problems above the important ingredient is a *group of symmetries*. In the first case this is a cyclic group of rotations of the table, in the second one the group of rotations of a cube, in the Venus problem the group of rotations of a dodecahedron (can you describe these groups?). Finally, in the last problem the group of all rotations in $\mathbb{R}^3$ appears. In all cases the group acts on a vector space via linear operators, i.e. as we have a *linear representation of a group*. The main part of this course deals with representation of groups.

Linear algebra problems. Every standard course of linear algebra discusses the problem of classification of all matrices in a complex vector space up to equivalence. (Here A is equivalent to B if $A = XBX^{-1}$ for some invertible X.) Indeed, there exists some basis in $\mathbb{C}^n$, in which A has a canonical Jordan form. The following problem is less known.

Kronecker problem. Let V and W be finite-dimensional vector spaces over algebraically closed field k, A and $B : V \to W$ be two linear operators. Classify all pairs (A, B) up to the change of bases in V and W. In other words we have to classify pairs of matrices up to the following equivalence relation: (A, B) is equivalent to (C, D) if there are invertible square matrices X and Y such that

$$C = XAY,\ D = XBY.$$

Theorem 1.1. *There exist decompositions*

$$V = V_1 \oplus \cdots \oplus V_k,\ W = W_1 \oplus \cdots \oplus W_k,$$

such that $A(V_i) \subset W_i$, $B(V_i) \subset W_i$ *and for each* $i \leq k$ *there exist bases in* V_i *and* W_i *such that the matrices for* A *and* B *have one of the following forms (here* 1_n *denotes the identity matrix of size* n, J_n *the nilpotent Jordan block of size* n *):*

$$A = \binom{1_n}{0},\ B = \binom{0}{1_n},\ n \geq 0;$$
$$A = (1_n, 0),\ B = (0, 1_n),\ n \geq 0;$$
$$A = 1_n,\ B = J_n,\ n \geq 1;$$
$$A = t1_n + J_n,\ B = 1_n,\ n \geq 1,\ t \in k.$$

2. Representations of groups. Definition and examples.

Let k denote a field, V be a vector space over k. By $\mathrm{GL}(V)$ we denote the group of all invertible linear operators in V. If $\dim V = n$, then $\mathrm{GL}(V)$ is isomorphic to the group of invertible $n \times n$ matrices with entries in k.

A *(linear) representation* of a group G in V is a homomorphism

$$\rho : G \to \mathrm{GL}(V).$$

The dimension of V is called the *degree* or the *dimension* of a representation ρ and it may be infinite. For any $s \in G$ we denote by ρ_s the image of s in $\mathrm{GL}(V)$ and for any $v \in V$ we denote by $\rho_s v$ the image of v under the action of ρ_s. The following properties are obvious:

$$\rho_s \rho_t = \rho_{st},\ \rho_1 = \mathrm{Id},\ \rho_s^{-1} = \rho_{s^{-1}},\ \rho_s(xv + yw) = x\rho_s v + y\rho_s w.$$

Examples.

1. Let $G = \mathbb{Z}$ with operation $+$, $V = \mathbb{R}^2$, ρ_n is given by the matrix

$$\begin{matrix} 1 & n \\ 0 & 1 \end{matrix}$$

for $n \in \mathbb{Z}$.

2. *Permutation representation.* Let $G = S_n$, $V = k^n$. For each $s \in S_n$ put

$$\rho_s(x_1, \dots, x_n) = \left(x_{s(1)}, \dots, x_{s(n)}\right).$$

3. *Trivial representation.* For any group G the trivial representation is the homomorphism $\rho : G \to k^*$ such that $\rho_s = 1$ for all $s \in G$.

4. Let G be a group and

$$\mathcal{F}(G) = \{f : G \to k\}$$

be the space of functions on G with values in k. For any $s \in G$, $f \in \mathcal{F}(G)$ and $t \in G$ let

$$\rho_s f(t) = f(ts).$$

Then $\rho : G \to \mathrm{GL}(\mathcal{F}(G))$ is a linear representation.

5. *Regular representation.* Recall that the *group algebra* $k(G)$ is the vector space of all finite linear combinations $\sum c_g g$, $c_g \in k$ with natural multiplication. The regular representation $R : G \to \mathrm{GL}(k(G))$ is defined in the following way

$$R_s\left(\sum c_g g\right) = \sum c_g s g.$$

Two representations $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ are *equivalent* (or *isomorphic*) if there exists an isomorphism $T : V \to W$ such that for all $s \in G$

$$T \circ \rho_s = \sigma_s \circ T.$$

Example. If G is finite then the representations in examples 4 and 5 are equivalent. Indeed, define $T : \mathcal{F}(G) \to k(G)$ by the formula

$$T(f) = \sum_{g \in G} f(g)\, g^{-1}.$$

Then for any $f \in \mathcal{F}(G)$ we have

$$T(\rho_s f) = \sum_{g \in G} \rho_s f(g)\, g^{-1} = \sum_{g \in G} f(gs)\, g^{-1} = \sum_{h \in G} f(h)\, s h^{-1} = R_s(Tf).$$

3. Operations with representations

Restriction on a subgroup: Let H be a subgroup of G. For any $\rho : G \to \mathrm{GL}(V)$ we denote by $\mathrm{Res}_H\, \rho$ the restriction of ρ on H.

Lift. Let $p : G \to H$ be a homomorphism of groups. For every representation $\rho : H \to \mathrm{GL}(V)$, $\rho \circ p : G \to \mathrm{GL}(V)$ is also a representation. We often use this construction in case when $H = G/N$ is a quotient group and p is the natural projection.

Direct sum. If we have two representations $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$, then we can define $\rho \oplus \sigma : G \to \mathrm{GL}(V \oplus W)$ by the formula

$$(\rho \oplus \sigma)_s (v, w) = (\rho_s v, \sigma_s w).$$

Tensor product. The tensor product of $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ is defined by the formula

$$(\rho \otimes \sigma)_s\, v \otimes w = \rho_s v \otimes \sigma_s w.$$

Exterior tensor product. Let $\rho : G \to \mathrm{GL}(V)$ and $\sigma : H \to \mathrm{GL}(W)$ be representations of two different groups, then their exterior product $\rho \boxtimes \sigma : G \times H \to \mathrm{GL}(V \otimes W)$ is defined by

$$(\rho \boxtimes \sigma)_{(s,t)}\, v \otimes w = \rho_s v \otimes \sigma_t w.$$

If $\delta : G \to G \times G$ is the diagonal embedding, then

$$\rho \otimes \sigma = (\rho \boxtimes \sigma) \circ \delta.$$

Dual representation. For any representation $\rho : G \to \mathrm{GL}(V)$ one can define the dual representation $\rho^* : G \to \mathrm{GL}(V^*)$ by the formula

$$< \rho_s^* \varphi, v > = < \varphi, \rho_s^{-1} v >$$

for any $v \in V, \varphi \in V^*$. Here $<,>$ denotes the natural pairing between V and V^*.

More generally, if $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ are two representations, then one can naturally define the representation τ of G in $\mathrm{Hom}_k(V, W)$ by the formula

$$\tau_s \varphi = \sigma_s \circ \varphi \circ \rho_s^{-1},\ s \in G,\ \varphi \in \mathrm{Hom}_k(V, W).$$

4. Invariant subspaces and irreducibility

Given a representation $\rho : G \to \mathrm{GL}(V)$. A subspace $W \subset V$ is called *invariant* if $\rho_s(W) \subset W$ for any $s \in G$. One can define naturally the subrepresentation

$$\rho^W : G \to \mathrm{GL}(W)$$

and the quotient representation

$$\sigma : G \to \mathrm{GL}(V/W).$$

Example. Let $\rho : S_n \to \mathrm{GL}(k^n)$ be the permutation representation, then

$$W = \{(x_1, \dots, x_n) \mid x_1 = x_2 = \dots = x_n\}$$

and

$$W' = \{(x_1, \dots, x_n) \mid x_1 + x_2 + \dots + x_n = 0\}$$

are invariant subspaces.

Theorem 4.1. *(Maschke) Let G be a finite group and* $\mathrm{char}\, k$ *do not divide $|G|$. Let $\rho : G \to \mathrm{GL}(V)$ be a representation and W be an invariant subspace. Then there exists another invariant subspace W' such that $V = W \oplus W'$.*

Proof. Let W'' be a subspace (not invariant) such that $V = W \oplus W''$. Let $P : V \to V$ be the linear operator such that $P_{|W} = \mathrm{Id}$ and $P(W'') = 0$. Then $P^2 = P$. Such operator is called a *projector*. Let

$$\bar{P} = \frac{1}{|G|} \sum_{g \in G} \rho_g \circ P \circ \rho_g^{-1}.$$

Check that $\rho_s \circ \bar{P} \circ \rho_s^{-1} = \bar{P}$, and hence $\rho_s \circ \bar{P} = \bar{P} \circ \rho_s$ for any $s \in G$. Check also that $\bar{P}_{|W} = \mathrm{Id}$ and $\mathrm{Im}\, \bar{P} = W$. Hence $\bar{P}^2 = \bar{P}$.

Let $W' = \mathrm{Ker}\, \bar{P}$. First, we claim that W' is invariant. Indeed, let $w \in W'$, then $\bar{P}(\rho_s w) = \rho_s \left(\bar{P} w\right) = 0$, hence $\rho_s w \in \mathrm{Ker}\, \bar{P} = W'$.

Now we prove that $V = W \oplus W'$. Indeed, $W \cap W' = 0$, since $\bar{P}_{|W} = Id$. On the other hand, for any $v \in V$, we have $w = \bar{P} v \in W$ and $w' = v - \bar{P} v \in W'$. Thus, $v = w + w'$, and therefore $V = W + W'$. □

In the previous example $V = W \oplus W'$ if $\operatorname{char} k$ does not divide n. Otherwise, $W \subset W'$, and the theorem is not true.

If G is infinite group, theorem is not true. Consider the representation of $\mathbb{Z}$ in $\mathbb{R}^2$ from Example 1. It has the unique one-dimensional invariant subspace, therefore $\mathbb{R}^2$ does not split into a direct sum of two invariant subspaces.

A representation is called *irreducible* if it does not contain a proper non-zero invariant subspace.

Exercise. Show that if $\operatorname{char} k$ does not divide n, then the subrepresentation W' of the permutation representation is irreducible.

Lemma 4.2. *Let G be a finite group, $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation. Then $\dim V \leq |G|$.*

Proof. Take any non-zero $v \in V$, then the set $\{\rho_s v\}_{s \in G}$ spans an invariant subspace which must coincide with V. Hence $\dim V \leq |G|$. □

Example. Let $\rho : G \to \mathrm{GL}(V)$. We claim that $\rho \otimes \rho$ is irreducible if and only if $\dim V = 1$. Indeed the subspaces $S^2 V, \Lambda^2 V \subset V \otimes V$ are invariant and $\Lambda^2 V = \{0\}$ only in case when $\dim V = 1$.

A representation is called *completely reducible* if it splits into a direct sum of irreducible subrepresentations.

Corollary 4.3. *Let G be a finite group and k be a field such that $\operatorname{char} k$ does not divide $|G|$. Then every finite-dimensional representation of G is completely reducible.*

Proof. By induction on $\dim V$. □

5. Schur's Lemma

For any two representations $\rho : G \to \mathrm{GL}(V)$, $\sigma : G \to \mathrm{GL}(W)$ let

$$\mathrm{Hom}_G(V, W) = \{T \in \mathrm{Hom}_k(V, W) \mid \sigma_s \circ T = T \circ \rho_s, s \in G\}.$$

An operator $T \in \mathrm{Hom}_G(V, W)$ is called an *intertwining* operator. It is clear that $\mathrm{Hom}_G(V, W)$ is a vector space. Moreover, if $\rho = \sigma$, then $\mathrm{Hom}_G(V, V) = \mathrm{End}_G(V)$ is closed under operation of composition, and therefore it is a k-algebra.

Lemma 5.1. *Let $T \in \mathrm{Hom}_G(V, W)$, then $\operatorname{Ker} T$ and $\operatorname{Im} T$ are invariant subspaces.*

Proof. Let $v \in \operatorname{Ker} T$, then $T(\rho_s v) = \rho_s(Tv) = 0$, hence $\rho_s v \in \operatorname{Ker} T$.

Let $w \in \operatorname{Im} T$. Then $w = Tv$ for some $v \in V$ and $\rho_s w = \rho_s(Tv) = T(\rho_s v) \in \operatorname{Im} T$. □

Corollary 5.2. *(Schur's lemma) Let $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ be irreducible representations of G, then any $T \in \mathrm{Hom}_G(V, W)$ is either isomorphism or zero.*

Proof. Since both V and W do not have proper invariant subspaces, then either $\operatorname{Im} T = W$, $\operatorname{Ker} T = \{0\}$ or $\operatorname{Im} T = \{0\}$. □

Corollary 5.3. *If $\rho : G \to \mathrm{GL}(V)$ is irreducible then $\mathrm{End}_G(V)$ is a division ring. If the field k is algebraically closed and V is finite-dimensional, then $\mathrm{End}_G(V) = k\,\mathrm{Id}$.*

Proof. The first assertion follows immediately from the Corollary 5.2. To prove the second, let $T \in \mathrm{End}_G(V)$ and $T \neq 0$. Then T is invertible. Let λ be an eigenvalue of T and $S = T - \lambda Id$. Since $S \in \mathrm{End}_G(V)$ and $\mathrm{Ker}\, S \neq \{0\}$, by Corollary 5.2, $S = 0$. Thus, $T = \lambda\,\mathrm{Id}$. □

Corollary 5.4. *Let G be an abelian group, $\rho : G \to \mathrm{GL}(V)$ be an irreducible finite-dimensional representation of G over algebraically closed field k. Then $\dim V = 1$.*

Irreducible representations of a finite cyclic group over $\mathbb{C}$. Let G be a cyclic group of order n and g be a generator. By Corollary 5.4 every irreducible representation of G is one-dimensional. Thus, we have to classify homomorphisms $\rho : G \to \mathbb{C}^*$. Let $\rho_g = \varepsilon$. Then clearly ε is an n-th root of 1. Therefore we have exactly n non-equivalent irreducible representations.

Irreducible representations of a finite abelian group over $\mathbb{C}$. Any finite abelian group is a direct product $G_1 \times \cdots \times G_k$ of cyclic groups. Let g_i be a generator of G_i. Then any irreducible $\rho : G \to \mathbb{C}^*$ is determined by its values $\rho_{g_i} = \varepsilon_i$, where $\varepsilon_i^{|G_i|} = 1$. Hence the number of isomorphism classes of irreducible representations of G equals $|G|$.

Remark 5.5. It is not difficult to see that the set of one-dimensional representations of G is a group with respect to the operation of tensor product. In case when G is finite and abelian and k is algebraically closed, all irreducible representations are one dimensional and form a group. We denote this group by $G^\vee$. As easily follows from above $G^\vee \cong G$ when $k = \mathbb{C}$, however this isomorphism is not canonical.

Here is another application of Schur's Lemma.

Theorem 5.6. *Let $\rho \cong \rho_1 \oplus \cdots \oplus \rho_k \cong \sigma_1 \oplus \cdots \oplus \sigma_m$, where ρ_i, σ_j are irreducible. Then $m = k$ and there exists $s \in S_k$ such that $\rho_j \cong \sigma_{s(j)}$.*

Proof. Let V be the space of a representation ρ. There are two decompositions of V into the direct sum of irreducible decomposable subspaces

$$V = V_1 \oplus \cdots \oplus V_k = W_1 \oplus \cdots \oplus W_m.$$

Let $p_i : V \to W_i$ be the projection which maps W_j to zero for $j \neq i$, $q_j : V_j \to V$ be the embedding. Then $p_i \in \mathrm{Hom}_G(V, W_i)$ and $q_j \in \mathrm{Hom}_G(V_j, W)$. The map

$$F = \sum_{i=1}^{m} \sum_{j=1}^{k} p_i \circ q_j : \oplus_{j=1}^{k} V_j \to \oplus_{i=1}^{m} W_i$$

is an isomorphism. There exists i such that $p_i \circ q_1 \neq 0$. (Otherwise $F(V_1) = 0$ which is impossible.) Put $s(1) = i$ and note that $p_i \circ q_1$ is an isomorphism by Schur's Lemma. We continue inductively. For each j there exists i such that $p_i \circ q_j$ is an isomorphism and $i \neq s(r)$ for any $r < j$. (Indeed, otherwise $F(V_1 \oplus \cdots \oplus V_j) \subset W_{s(1)} \oplus \cdots \oplus W_{s(j-1)}$ which is impossible because F is an isomorphism.) We put $i = s(j)$. Thus, we can construct an injective map

$$s : \{1, \ldots, k\} \to \{1, \ldots, m\}$$

such that $\rho_j \cong \sigma_{s(j)}$. In particular, $k \leq m$. But by exchanging ρ_i and σ_j we can prove that $m \leq k$. Hence $k = m$ and s is a permutation. □

PROBLEM SET # 1
MATH 252

Due September 9.

1. Classify irreducible representations of $\mathbb{Z}$ over $\mathbb{C}$.

2. Classify one-dimensional represenations of S_n over any field k such that $\operatorname{char} k \neq 2$.

3. Let ρ be an irreducible representation of G and σ be an irreducible representation of H. Is it always true that the exterior tensor product of ρ and σ is an irreducible representation of $G \times H$?

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REPRESENTATION THEORY
WEEK2

VERA SERGANOVA

1. Characters

For any finite-dimensional representation $\rho : G \to \mathrm{GL}(V)$ its *character* is a function $\chi_\rho : G \to k$ defined by

$$\chi_\rho(s) = \operatorname{tr} \rho_s.$$

It is easy to see that characters have the following properties

(1) $\chi_\rho(1) = \dim \rho$;
(2) if $\rho \cong \sigma$, then $\chi_\rho = \chi_\sigma$;
(3) $\chi_{\rho \oplus \sigma} = \chi_\rho + \chi_\sigma$;
(4) $\chi_{\rho \otimes \sigma} = \chi_\rho \chi_\sigma$;
(5) $\chi_{\rho^*}(s) = \chi_\rho(s^{-1})$;
(6) $\chi_\rho(sts^{-1}) = \chi_\rho(t)$.

Example 1. If R is a regular representation, then $\chi_R(s) = 0$ for any $s \neq 1$ and $\chi_R(1) = |G|$.

Example 2. Let $\rho : G \to \mathrm{GL}(V)$ be a representation. Recall that $\rho \otimes \rho = \mathrm{sym} \oplus \mathrm{alt}$, where $\mathrm{alt}: G \to \mathrm{GL}(\Lambda^2 V)$ and $\mathrm{sym}: G \to \mathrm{GL}(S^2 V)$. Let us calculate χ_{sym} and χ_{alt}. For each $s \in G$ let $\lambda_1, \dots, \lambda_n$ be eigenvalues of ρ_s taken with multiplicities. Then the eigenvalues of alt_s are $\lambda_i \lambda_j$ for all $i < j$ and the eigenvalues of sym_s are $\lambda_i \lambda_j$ for $i \leq j$. Hence

$$\chi_{\mathrm{sym}}(s) = \sum_{i \leq j} \lambda_i \lambda_j, \ \chi_{\mathrm{alt}}(s) = \sum_{i < j} \lambda_i \lambda_j,$$

and therefore

$$\chi_{\mathrm{sym}}(s) - \chi_{\mathrm{alt}}(s) = \sum_i \lambda_i^2 = \operatorname{tr} \rho_{s^2} = \chi_\rho\left(s^2\right).$$

On the other hand by properties (3) and (4)

$$\chi_{\mathrm{sym}}(s) + \chi_{\mathrm{alt}}(s) = \chi_{\rho \otimes \rho}(s) = \chi_\rho^2(s).$$

Thus, we get

$$\chi_{\mathrm{sym}}(s) = \frac{\chi_\rho^2(s) + \chi_\rho(s^2)}{2}, \ \chi_{\mathrm{alt}}(s) = \frac{\chi_\rho^2(s) - \chi_\rho(s^2)}{2}.$$

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Starting from this point we assume that G is finite and k is algebraically closed of characteristic 0.

Introduce the non-degenerate symmetric bilinear form on the space of functions $\mathcal{F}(G)$ by the formula

$$(f,g)=\frac{1}{|G|}\sum_{s\in G} f\left(s^{-1}\right) g\left(s\right).$$

Theorem 1.1. *Let ρ, σ be irreducible. If ρ and σ are not isomorphic, then $(\chi_\rho, \chi_\sigma) = 0$. If ρ and σ are isomorphic, then $(\chi_\rho, \chi_\sigma) = 1$.*

Proof. Let V be the space of the representation ρ and W be the space of σ. Denote $n = \dim V$, $m = \dim W$. Choose a basis $v_1, \dots, v_n$ in V, $w_1, \dots, w_m$ in W. Define $P(i,j) : W \to W$ by

$$P(i,j)\, v_k = \delta_{jk} w_i.$$

Lemma 1.2. *For any $T \in \operatorname{Hom}_k(V,W)$ let*

$$\bar{T}=\frac{1}{|G|}\sum_{s\in G}\sigma_s\circ T\circ\rho_{s^{-1}}.$$

Then $\bar{T} \in \operatorname{Hom}_G(V,W)$. If $V = W$, then $\operatorname{tr} T = \operatorname{tr} \bar{T}$.

Proof. Direct calculations. □

For any $T \in \operatorname{Hom}(V,W)$ let T_{kl} denote the corresponding matrix entry. For example, $P(i,j)_{kl} = \delta_{ik}\delta_{jl}$. Then

$$\bar{P}(i,j)_{kl}=\frac{1}{|G|}\sum_{s\in G}(\sigma_s)_{ki}\,(\rho_{s^{-1}})_{jl}\,.$$

If σ and ρ are not isomorphic, then by Schur's Lemma

$$\bar{P}(i,j)_{kl}=0$$

for all i,j,k,l. In particular, $\bar{P}(i,j)_{ij} = 0$ and therefore

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\bar{P}(i,j)_{ij}=\frac{1}{|G|}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{s\in G}(\sigma_s)_{ii}\,(\rho_{s^{-1}})_{jj}=0.$$

But

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(\sigma_s)_{ii}\,(\rho_{s^{-1}})_{jj}=\chi_\sigma(s)\,\chi_\rho\left(s^{-1}\right).$$

Hence

$$\frac{1}{|G|}\sum_{s\in G}\chi_\sigma(s)\,\chi_\rho\left(s^{-1}\right)=(\chi_\rho,\chi_\sigma)=0.$$

Let now $\rho \cong \sigma$. The by Property (2), we may assume $\rho = \sigma$. Then by Schur's Lemma

$$\bar{P}(i,j)=\lambda\,\mathrm{Id}\,.$$

Since $\operatorname{tr} \bar{P}(i,j) = \operatorname{tr} P(i,j) = \delta_{ij}$, we have

$$\bar{P}(i,j) = \frac{\delta_{ij}}{n} \operatorname{Id}.$$

Then

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \bar{P}(i,j)_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\delta_{ij}}{n} = 1,$$

which implies $(\chi_\rho, \chi_\rho) = 1$. □

Corollary 1.3. *Let $\rho = m_1\rho_1 \oplus \cdots \oplus m_l\rho_l$ be decomposition into the sum of irreducibles. Then $m_i = (\chi_\rho, \chi_{\rho_i})$.*

The number m_i is called the *multiplicity* of an irreducible representation ρ_i in ρ.

Corollary 1.4. *A representation ρ is irreducible iff $(\chi_\rho, \chi_\rho) = 1$.*

Corollary 1.5. *Every irreducible representation ρ appears in the regular representation with multiplicity $\dim \rho$.*

Proof.

$$(\chi_\rho, \chi_R) = \frac{1}{|G|} \chi_\rho(1) \chi_R(1) = \dim \rho.$$

□

Corollary 1.6. *Let $\rho_1, \dots, \rho_l$ be all (up to isomorphism) irreducible representations of G and $n_i = \dim \rho_i$. Then*

$$n_1^2 + \cdots + n_l^2 = |G|.$$

Example 3. Let G act on a finite set X and

$$k(X) = \left\{ \sum_{x \in X} b_x x \mid b_x \in k \right\}.$$

Define $\rho : G \to \operatorname{GL}(k(X))$ by

$$\rho_s \sum_{x \in X} b_x x = \sum_{x \in X} b_x s(x).$$

It is easy to check that ρ is a representation and

$$\chi_\rho(s) = |\{x \in X \mid s(x) = x\}|.$$

Clearly, ρ contains a trivial subrepresentation. To find the multiplicity of a trivial representation in ρ we have to calculate $(1, \chi_\rho)$.

$$(1, \chi_\rho) = \frac{1}{|G|} \sum_{s \in G} \chi_\rho(s) = \frac{1}{|G|} \sum_{s \in G} \sum_{s(x)=x} 1 = \frac{1}{|G|} \sum_{x \in X} \sum_{s \in G_x} 1 = \frac{1}{|G|} \sum_{x \in X} |G_x|,$$

where

$$G_x = \{s \in G \mid s(x) = x\}.$$

Let $X = X_1 \cup \dots \cup X_m$ be the disjoint union of orbits. Then $|G_x| = \frac{|G|}{|X_i|}$ for each $x \in X_i$ and therefore

$$(1, \chi_\rho) = \frac{1}{|G|} \sum_{i=1}^{m} \sum_{x \in X_i} \frac{|G|}{|X_i|} = m.$$

Now let us evaluate (χ_ρ, χ_ρ).

$$(\chi_\rho, \chi_\rho) = \frac{1}{|G|} \sum_{s \in G} \left(\sum_{s(x)=x} 1 \right)^2 = \frac{1}{|G|} \sum_{s \in G} \sum_{s(x)=x, s(y)=y} 1 = \frac{1}{|G|} \sum_{(x,y) \in X \times X} |G_x \cap G_y|.$$

Let σ be the representation associated with the action of G on $X \times X$. Then the last formula implies

$$(\chi_\rho, \chi_\rho) = (1, \chi_\sigma).$$

Thus, ρ is irreducible iff $|X| = 1$, and ρ has two irreducible components iff the action of G on $X \times X$ with removed diagonal is transitive or $|X| = 2$.

Let

$$\mathcal{C}(G) = \left\{ f \in \mathcal{F}(G) \mid f\left(sts^{-1}\right) = f(t) \right\}.$$

It is easy to check that the restriction of $(,)$ on $\mathcal{C}(G)$ is non-degenerate.

Theorem 1.7. *The characters of irreducible representations of G form an orthonormal basis in $\mathcal{C}(G)$.*

Proof. We have to show that if $f \in \mathcal{C}(G)$ and $(f, \chi_\rho) = 0$ for any irreducible ρ, then $f = 0$. The following lemma is straightforward.

Lemma 1.8. *Let $\rho : G \to \mathrm{GL}(V)$ be a representation, $f \in \mathcal{C}(G)$ and*

$$T = \frac{1}{|G|} \sum_{s \in G} f\left(s^{-1}\right) \rho_s.$$

Then $T \in \mathrm{End}_G V$ and $\mathrm{tr}\, T = (f, \chi_\rho)$.

Thus, for any irreducible ρ we have

$$\frac{1}{|G|} \sum_{s \in G} f\left(s^{-1}\right) \rho_s = 0. \tag{1.1}$$

But then the same is true for any representation ρ, since any representation is a direct sum of irreducibles. Apply (**?**equ1**?**) for the case when $\rho = R$ is the regular representation. Then

$$\frac{1}{|G|} \sum_{s \in G} f\left(s^{-1}\right) R_s 1 = \frac{1}{|G|} \sum_{s \in G} f\left(s^{-1}\right) s = 0.$$

Hence $f\left(s^{-1}\right) = 0$ for all $s \in G$, i.e. $f = 0$ □

Corollary 1.9. *The number of isomorphism classes of irreducible representations equal the number of the conjugacy classes.*

PROBLEM SET # 2
MATH 252

Due September 16.

In this problem set the field is algebraically closed and has zero characteristic, G is finite and representations are finite-dimensional.

1. Show that the statement of Problem 3 in the first problem set is correct under above assumptions.

2. Let $\rho : G \to \mathrm{GL}(V)$ be a representation. Show that each irreducible subrepresentation of V has multiplicity 1 iff $\mathrm{End}_G V$ is a commutative ring.

3. Let G be the subgroup of quaternions of 8 elements, that contains $\pm 1, \pm i, \pm j, \pm k$ with relations $i^2 = j^2 = k^2 = -1$, $ij = k$, $jk = i$, $ki = j$, $ij = -ji$, $ik = -ki$, $jk = -kj$. Classify irreducible representations of G over $\mathbb{C}$.

Date: September 8, 2005.

REPRESENTATION THEORY. WEEK 3

VERA SERGANOVA

1. Examples.

Example 1. Let $G = S_3$. There are three conjugacy classes in G, which we denote by some element in a class:1,(12),(123). Therefore there are three irreducible representations, denote their characters by χ_1, χ_2 and χ_3. It is not difficult to see that we have the following table of characters

	1	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

The characters of one-dimensional representations are given in the first and the second row, the last character χ_3 can be obtained by using the identity

$$\chi_{\text{perm}} = \chi_1 + \chi_3, \tag{1.1}$$

where χ_{perm} stands for the character of the permutation representation.

Example 2. Let $G = S_4$. In this case we have the following character table (in the first row we write the number of elements in each conjugacy class).

	1	6	8	3	6
	1	(12)	(123)	(12) (34)	(1234)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	3	1	0	-1	-1
χ_4	3	-1	0	-1	1
χ_5	2	0	-1	2	0

First two rows are characters of one-dimensional representations. The third can be obtained again from (1.1), $\chi_4 = \chi_2\chi_3$, the corresponding representation is obtained as the tensor product $\rho_4 = \rho_2 \otimes \rho_3$. The last character can be found from the orthogonality relation. Alternative way to describe ρ_5 is to consider S_4/V_4, where

$$V_4 = \{1, (12)\,(34)\,, (13)\,(24)\,, (14)\,(23)\}$$

Date: September 16, 2005.

is the Klein subgroup. Observe that $S_4/V_4 \cong S_3$, and therefore the two-dimensional representation σ of S_3 can be extended to the representation of S_4 by the formula

$$\rho_5 = \sigma \circ p,$$

where $p : S_4 \to S_3$ is the natural projection.

Solution of the marcian problem. Recall that S_4 is isomorphic to the group of rotations of a cube. Hence it acts on the set of faces F, and therefore we have a representation

$$\rho\colon S_4 \to \mathrm{GL}\left(\mathbb{C}\left(F\right)\right).$$

It is not difficult to calculate χ_ρ using the formula

$$\chi_\rho\left(s\right) = \left|\left\{x \in F \mid s\left(x\right) = x\right\}\right|.$$

We obtain

$$\chi_\rho\left(1\right) = 6,\ \chi_\rho\left(\left(12\right)\right) = \chi_\rho\left(\left(123\right)\right) = 0,\ \chi_\rho\left(\left(12\right)\left(34\right)\right) = \chi_\rho\left(\left(1234\right)\right) = 2.$$

Furthermore,

$$(\chi_\rho, \chi_1) = (\chi_\rho, \chi_4) = (\chi_\rho, \chi_5) = 1,\ \ (\chi_\rho, \chi_2) = (\chi_\rho, \chi_3) = 0.$$

Hence $\chi_\rho = \chi_1 + \chi_4 + \chi_5$, and $\mathbb{C}\left(F\right) = W_1 \oplus W_2 \oplus W_3$ the sum of three invariant subspaces. The intertwining operator $T : \mathbb{C}\left(F\right) \to \mathbb{C}\left(F\right)$ of food redistribution must be a scalar operator on each W_i by Schur's Lemma. Note that

$$W_1 = \left\{\sum_{x\in F} ax \mid a \in \mathbb{C}\right\},$$

$$W_2 = \left\{\sum_{x\in F} a_x x \mid a_x = -a_{x_{\mathrm{op}}}\right\},$$

$$W_3 = \left\{\sum_{x\in F} a_x x \mid \sum a_x = 0, a_x = a_{x_{\mathrm{op}}}\right\},$$

where x_{op} denotes the face opposite to the face x. A simple calculation shows that $T_{|W_1} = \mathrm{Id}$, $T_{|W_2} = 0$, $T_{|W_3} = -\frac{1}{2}Id$. Therefore $T^n\left(v\right)$ approaches a vector in W_1 as $n \to \infty$, and eventually everybody will have the same amount of food.

Example 3. Now let $G = A_5$. There are 5 irreducible representation of G over $\mathbb{C}$. Here is the character table

	1	20	15	12	12
	1	(123)	(12) (34)	(12345)	(12354)
χ_1	1	1	1	1	1
χ_2	4	1	0	-1	-1
χ_3	5	-1	1	0	0
χ_4	3	0	-1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
χ_5	3	0	-1	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$

To obtain χ_2 use the permutation representation and (1.1) again. Let χ_{sym} and χ_{alt} be the characters of the second symmetric and the second exterior powers of ρ_2 respectively. Then we obtain

	1	(123)	(12) (34)	(12345)	(12354)
χ_{sym}	10	1	2	0	0
χ_{alt}	6	0	−2	1	1

It is easy to check that

$$(\chi_{\text{sym}}, \chi_{\text{sym}}) = 3,\ (\chi_{\text{sym}}, \chi_1) = (\chi_{\text{sym}}, \chi_2) = 1.$$

Therefore

$$\chi_3 = \chi_{\text{sym}} - \chi_1 - \chi_2$$

is the character of an irreducible representation.

Furthermore,

$$(\chi_{\text{alt}}, \chi_{\text{alt}}) = 2,\ (\chi_{\text{alt}}, \chi_1) = (\chi_{\text{alt}}, \chi_2) = (\chi_{\text{alt}}, \chi_3) = 0.$$

Therefore $\chi_{\text{alt}} = \chi_4 + \chi_5$ is the sum of two irreducible characters. First we find the dimensions of ρ_4 and ρ_5 using

$$1^2 + 4^2 + 5^2 + n_4^2 + n_5^2 = 60.$$

We obtain $n_4 = n_5 = 3$. The equations

$$(\chi_4, \chi_1 + \chi_2) = 0,\ (\chi_4, \chi_3) = 0$$

imply

$$\chi_4((123)) = 0,\ \chi_4((12)(34)) = -1.$$

The same argument shows

$$\chi_5((123)) = 0,\ \chi_5((12)(34)) = -1.$$

Finally denote

$$x = \chi_4((12345)),\ y = \chi_4((12354))$$

and write down the equation $(\chi_4, \chi_4) = 1$. It is

$$\frac{1}{60}\left(9 + 15 + 12x^2 + 12y^2\right) = 1,$$

or

$$x^2 + y^2 = 3. \tag{1.2}$$

On the other hand, $(\chi_4, \chi_1) = 0$, that gives

$$3 - 15 + 12(x + y) = 0,$$

or

$$x + y = 1. \tag{1.3}$$

One can solve (1.2) and (1.3)

$$x = \frac{1+\sqrt{5}}{2}, y = \frac{1-\sqrt{5}}{2}.$$

The second solution

$$x = \frac{1-\sqrt{5}}{2}, y = \frac{1+\sqrt{5}}{2}$$

will give the last character χ_5.

2. Modules

Let R be a ring, usually we assume that $1 \in R$. An abelian group M is called a *(left) R-module* if there is a map $\alpha\colon R \times M \to M$, (we write $\alpha(a,m) = am$) satisfying

(1) $(ab)\,m = a\,(bm)$;
(2) $1m = m$;
(3) $(a+b)\,m = am + bm$;
(4) $a\,(m+n) = am + an$.

Example 1. If R is a field then any R-module M is a vector space over R.

Example 2. If $R = k\,(G)$ is a group algebra, then every R-module defines the representation $\rho\colon \to \mathrm{GL}\,(V)$ by the formula

$$\rho_s v = sv$$

for any $s \in G \subset k\,(G)$, $v \in V$. Conversely, every representation $\rho\colon G \to \mathrm{GL}\,(V)$ in a vector space V over k defines on V a $k\,(G)$-module structure by

$$\left(\sum_{s\in G} a_s s\right) v = \sum_{s\in G} a_s \rho_s v.$$

Thus, representations of G over k are $k\,(G)$-modules.

A *submodule* is a subgroup invariant under R-action. If $N \subset M$ is a submodule then the quotient M/N has the natural R-module structure. A module M is *simple* or (*irreducible*) if any submodule is either zero or M. A sum and an intersection of submodules is a submodule.

Example 3. If R is an arbitrary ring, then R is a left module over itself, where the action is given by the left multiplication. Submodules are left ideals.

For any two R-modules M and N one can define an abelian group $\mathrm{Hom}_R\,(M,N)$ and a ring $\mathrm{End}_R\,(M)$ in the manner similar to the group case. Schur's Lemma holds for simple modules in the following form.

Lemma 2.1. *Let M and N be simple modules and $\varphi \in \mathrm{Hom}_R\,(M,N)$, then either φ is an isomorphism or $\varphi = 0$. For a simple module M, $\mathrm{End}_R\,(M)$ is a division ring.*

Recall that for every ring R one defines R^{op} as the same abelian group with new multiplication $*$ given by

$$a * b = ba.$$

Lemma 2.2. *The ring* $\mathrm{End}_R(R)$ *is isomorphic to* R^{op}.

Proof. For each $a \in R$ define $\varphi_a \in \mathrm{End}(R)$ by the formula

$$\varphi_a(x) = xa.$$

It is easy to check that $\varphi_a \in \mathrm{End}_R(R)$ and $\varphi_{ba} = \varphi_a \circ \varphi_b$. Hence we constructed a homomorphism $\varphi : R^{\mathrm{op}} \to \mathrm{End}_R(R)$. To prove that φ is injective let $\varphi_a = \varphi_b$. Then $\varphi_a(1) = \varphi_b(1)$, i.e. $a = b$. To prove surjectivity of φ, note that for any $\gamma \in \mathrm{End}_R(R)$ one has

$$\gamma(x) = \gamma(x1) = x\gamma(1).$$

Therefore $\gamma = \varphi_{\gamma(1)}$. □

Lemma 2.3. *Let* $\rho_i : G \to \mathrm{GL}(V_i)$, $i = 1, \dots, l$ *be pairwise non-isomorphic irreducible representations of a group* G *over algebraically closed field* k, *and*

$$V = V_1^{\oplus m_1} \oplus \cdots \oplus V_l^{\oplus m_l}.$$

Then

$$\mathrm{End}_G(V) \cong \mathrm{End}_k(k^{m_1}) \times \cdots \times \mathrm{End}_k(k^{m_l}).$$

Proof. First, note that the Schur's Lemma implies that $\varphi\left(V_i^{\oplus m_i}\right) \subset V_i^{\oplus m_i}$ for any $\varphi \in \mathrm{End}_G(V)$, $i = 1, \dots, l$. Hence

$$\mathrm{End}_G(V) \cong \mathrm{End}_G\left(V_1^{\oplus m_1}\right) \times \cdots \times \mathrm{End}_G\left(V_l^{\oplus m_l}\right).$$

Therefore it suffices to prove the following

Lemma 2.4. *For an irreducible representation of* G *in* W

$$\mathrm{End}_G\left(W^{\oplus m}\right) \cong \mathrm{End}_k(k^m).$$

Proof. Let $p_i : W^{\oplus m} \to W$ denotes the projection onto the i-th component and $q_j : W \to W^{\oplus m}$ be the embedding of the j-th component. Let $\varphi \in \mathrm{End}_G(W^{\oplus m})$. Then by Schur's Lemma $p_i \circ \varphi \circ q_j = \varphi_{ij}\,\mathrm{Id}$ for some $\varphi_{ij} \in k$. Therefore we have the map $\Phi : \mathrm{End}_G(W^{\oplus m}) \to \mathrm{End}_k(k^m)$, (the latter is just the matrix ring) defined by $\Phi(\varphi) = (\varphi_{ij})$. Check yourself that Φ is an isomorphism. □

□

Theorem 2.5. *Let* k *be algebraically closed,* $\mathrm{char}\, k = 0$. *Then*

$$k(G) \cong \mathrm{End}_k(k^{n_1}) \times \cdots \times \mathrm{End}_k(k^{n_l}),$$

where $n_1, \dots, n_l$ *are dimensions of irreducible representations.*

Proof. By Lemma 2.2

$$\operatorname{End}_{k(G)}(k(G)) \cong k(G)^{\mathrm{op}} \cong k(G),$$

since $k(G)^{\mathrm{op}} \cong k(G)$ via $g \to g^{-1}$. On the other hand

$$k(G) = V_1^{\oplus n_1} \oplus \cdots \oplus V_l^{\oplus n_l},$$

since every irreducible $\rho_i : G \to \mathrm{GL}(V_i)$ appears with the multiplicity $n_i = \dim V_i$. Therefore Lemma 2.3 implies theorem. □

3. Finitely generated modules and Noetherian rings.

A module M is *finitely generated* if there exist $x_1, \ldots, x_n \in M$ such that $M = Rx_1 + \cdots + Rx_n$.

Lemma 3.1. *Let*

$$0 \to N \xrightarrow{q} M \xrightarrow{p} L \to 0$$

be an exact sequence of R-modules. If M is finitely generated, then L is finitely generated. If M and L are finitely generated, then M is finitely generated.

Proof. First assertion is obvious. For the second let

$$L = Rx_1 + \ldots Rx_n,\ N = Ry_1 + \ldots Ry_m,$$

then $M = Rp^{-1}(x_1) + \cdots + Rp^{-1}(x_n) + Rq(y_1) + \cdots + Rq(y_m)$. □

Lemma 3.2. *The following conditions on a ring R are equivalent*

1. *Every increasing chain of left ideals is finite, in other words for any sequence $I_1 \subset I_2 \subset \ldots$, $I_n = I_{n+1} = I_{n+2} = \ldots$ starting from some n;*
2. *Every left ideal is finitely generated R-module.*

Proof. (1)⇒ (2). Assume that some left ideal I is not finitely generated. Then there exists an infinite sequence of $x_n \in I$ such that

$$x_{n+1} \notin Rx_1 + \ldots Rx_n.$$

But then $I_n = Rx_1 + \ldots Rx_n$ form an infinite increasing chain of ideals.

(2)⇒ (1). Let $I_1 \subset I_2 \subset \ldots$ be an increasing chain of ideals . Let $I = \cup_n I_n$. Then $I = Rx_1 + \ldots Rx_s$, where $x_j \in I_{n_j}$. Let m be maximal among $n_1, \ldots, n_s$. Then $I = I_m$, and therefore the chain is finite. □

A ring satisfying the conditions of Lemma 3.2 is called *left Noetherian*.

Lemma 3.3. *Let R be a left Noetherian ring and M be a finitely generated R-module. Then every submodule of M is finitely generated.*

Proof. Let $M = Rx_1 + \dots Rx_n$, the there exists a surjective homomorphism $p : R \oplus \dots \oplus R \to M$, such that

$$p(r_1, \dots, r_n) = r_1 s_1 + \dots + r_n s_n.$$

As follows from the first part of Lemma 3.1, it suffices to prove the statement for a free module. It can be done by induction using the second part of Lemma 3.1. □

Let R be a commutative ring. An element $x \in R$ is called *integral over* $\mathbb{Z}$ if $x^n + a_{n-1}x + \dots + a_0 = 0$ for some $a_i \in \mathbb{Z}$. This condition is equivalent to the condition that $\mathbb{Z}[x] \subset R$ is finitely generated $\mathbb{Z}$-module. Complex numbers integral over $\mathbb{C}$ are usually called algebraic integers. Obviously, if a rational number z is algebraic integer, then $z \in \mathbb{Z}$.

Lemma 3.4. *The set of integral elements in a commutative ring R is a subring.*

Proof. If $\mathbb{Z}[x]$ and $\mathbb{Z}[y]$ are finitely generated over $\mathbb{Z}$, then $\mathbb{Z}[x, y]$ is also finitely generated. Let $s \in \mathbb{Z}[x, y]$, then $\mathbb{Z}[s]$ is finitely generated since $\mathbb{Z}$ is Noetherian ring and we can apply Lemma 3.3. □

4. The center of the group algebra $k(G)$

We have assumptions $\operatorname{char} k = 0$, $\bar{k} = k$, G is a finite group. Let $Z(G)$ denote the center of $k(G)$. It is obvious that

$$Z(G) = \left\{ \sum_{s \in G} f(s)\, s \mid f \in \mathcal{C}(G) \right\}.$$

On the other hand, by Theorem 2.5 we have

$$k(G) = \operatorname{End}_k(k^{n_1}) \times \dots \times \operatorname{End}_k(k^{n_l}).$$

Therefore $Z(G)$ is isomorphic to k^l as a commutative ring. Let e_i denote the identity element in $\operatorname{End}_k(k^{n_i})$. Then $e_1, \dots, e_l$ form a basis in $Z(G)$ and

$$e_i e_j = \delta_{ij} e_i,\ 1 = e_1 + \dots + e_l.$$

For an irreducible representation $\rho_j : G \to \mathrm{GL}(V_j)$ we have

$$\rho_j(e_i) = \delta_{ij} \operatorname{Id}. \tag{4.1}$$

Lemma 4.1. *If $\chi_i = \chi_{\rho_i}$, $n_i = \dim V_i$, then*

$$e_i = \frac{n_i}{|G|} \sum \chi_i(s^{-1})\, s. \tag{4.2}$$

Proof. We need to check (4.1). Since $\rho_j(e_i) \in \mathrm{End}_G(V_j)$, we have $\rho_j(e_i) = \lambda Id$. To find λ calculate

$$\operatorname{tr} \rho_j(e_i) = \frac{n_i}{|G|} \sum \chi_i\left(s^{-1}\right) \chi_j(s) = \frac{n_i}{|G|}(\chi_i, \chi_j) = \delta_{ij} n_i.$$

□

Lemma 4.2. *Define $\omega_i : Z(G) \to k$ by the formula*

$$\omega_i\left(\sum a_s s\right) = \frac{1}{n_i} \sum a_s \chi_i(s).$$

Then ω_i is a homomorphism of rings and

$$\omega = (\omega_1, \ldots, \omega_l) : Z(G) \to k^l$$

is an isomorphism.

Proof. Check that $\omega_i(e_j) = \delta_{ij}$ using again the orthogonality relation. □

Lemma 4.3. *Let $u = \sum a_s s \in Z(G)$. If all a_s are algebraic integers, then u is integral over $\mathbb{Z}$.*

Proof. Let $c \subset G$ be some conjugacy class and let

$$\delta_c = \sum_{s \in c} s.$$

If $c_1, \ldots, c_l$ are disjoint conjugacy classes, then clearly $\mathbb{Z}\delta_{c_1} + \cdots + \mathbb{Z}\delta_{c_l}$ is a subring in $Z(G)$. On the other hand, it is clearly a finitely generated $\mathbb{Z}$-module, and therefore every element of it is integral over $\mathbb{Z}$. But then for any set of algebraic integers $b_1, \ldots, b_l$ the element $\sum b_i \delta_{c_i}$ is integral over $\mathbb{Z}$, which proves Lemma. □

Theorem 4.4. *The dimension of an irreducible representation divides $|G|$.*

Proof. For every $s \in G$, $\chi_i(s)$ is an algebraic integer. Therefore by Lemma 4.3 $u = \sum_{s \in G} \chi_i(s^{-1}) s$ is integral over $\mathbb{Z}$. Hence $\omega_i(u)$ is an algebraic integer. But

$$\omega_i(u) = \frac{1}{n_i} \sum \chi_i(s) \chi_i\left(s^{-1}\right) = \frac{|G|}{n_i}(\chi_i, \chi_i) = \frac{|G|}{n_i}.$$

Therefore $\frac{|G|}{n_i} \in \mathbb{Z}$. □

Theorem 4.5. *Let Z be the center of G. The dimension n of an irreducible representation divides $\frac{|G|}{|Z|}$.*

Proof. Consider

$$\rho_m = \rho^{\boxtimes m} : G \times \cdots \times G \to \mathrm{GL}\left(V^{\otimes m}\right).$$

Then $\operatorname{Ker} \rho_m$ contains a subgroup

$$Z'_m = \{(z_1, \ldots, z_m) \in Z^m \mid z_1 z_2 \ldots z_m = 1\}.$$

If ρ is irreducible, then ρ_m is irreducible, and $\dim \rho_m = (\dim \rho)^m$ divides $|G^m/Z'_m| = \frac{|G|^m}{|Z|^{m-1}}$. Since this is true for any m, then $\dim \rho$ divides $\frac{|G|}{|Z|}$ (check yourself using prime factorization). $\square$

PROBLEM SET # 3
MATH 252

Due September 23.

1. Consider the action of the group A_5 on the faces of a dodecahedron. Decompose the corresponding representation of A_5 into a sum of irreducibles and solve the Venus problem by diagonalizing the intertwining operator.

2. Let D_n denote the dihedral group, which is the group of all symmetries of a regular n-gon. Classify irreducible representations of D_n over $\mathbb{C}$.

Date: September 15, 2005.

REPRESENTATION THEORY. WEEK 4

VERA SERGANOVA

1. Induced modules

Let $B \subset A$ be rings and M be a B-module. Then one can construct *induced* module $\operatorname{Ind}_B^A M = A \otimes_B M$ as the quotient of a free abelian group with generators from $A \times M$ by relations

$$(a_1 + a_2) \times m - a_1 \times m - a_2 \times m,\ a \times (m_1 + m_2) - a \times m_1 - a \times m_2,\ ab \times m - a \times bm,$$

and A acts on $A \otimes_B M$ by left multiplication. Note that $j : M \to A \otimes_B M$ defined by

$$j(m) = 1 \otimes m$$

is a homomorphism of B-modules.

Lemma 1.1. *Let N be an A-module, then for $\varphi \in \operatorname{Hom}_B(M, N)$ there exists a unique $\psi \in \operatorname{Hom}_A(A \otimes_B M, N)$ such that $\psi \circ j = \varphi$.*

Proof. Clearly, ψ must satisfy the relation

$$\psi(a \otimes m) = a\psi(1 \otimes m) = a\varphi(m).$$

It is trivial to check that ψ is well defined. □

Exercise. Prove that for any B-module M there exists a unique A-module satisfying the conditions of Lemma 1.1.

Corollary 1.2. *(Frobenius reciprocity.) For any B-module M and A-module N there is an isomorphism of abelian groups*

$$\operatorname{Hom}_B(M, N) \cong \operatorname{Hom}_A(A \otimes_B M, N).$$

Example. Let $k \subset F$ be a field extension. Then induction Ind_k^F is an exact functor from the category of vector spaces over k to the category of vector spaces over F, in the sense that the short exact sequence

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

becomes an exact sequence

$$0 \to F \otimes_k V_1 \otimes \to F \otimes_k V_2 \to F \otimes_k V_3 \to\ 0.$$

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In general, the latter property is not true. It is not difficult to see that induction is right exact, i.e. an exact sequence of B-modules

$$M \to N \to 0$$

induces an exact sequence of A-modules

$$A \otimes_B M \to A \otimes_B N \to 0.$$

But an exact sequence

$$0 \to M \to N$$

is not necessarily exact after induction.

Later we discuss general properties of induction but now we are going to study induction for the case of groups.

2. Induced representations for groups.

Let H be a subgroup of G and $\rho : H \to \mathrm{GL}(V)$ be a representation. Then the induced representation $\mathrm{Ind}_H^G \rho$ is by definition a $k(G)$-module

$$k(G) \otimes_{k(H)} V.$$

Lemma 2.1. *The dimension of* $\mathrm{Ind}_H^G \rho$ *equals the product of* $\dim \rho$ *and the index* $[G : H]$ *of* H. *More precisely, let* S *is a set of representatives of left cosets, i.e.*

$$G = \coprod_{s \in S} sH,$$

then

$$k(G) \otimes_{k(H)} V = \oplus_{s \in S} s \otimes V. \tag{2.1}$$

For any $t \in G$, $s \in S$ *there exist unique* $s' \in S$, $h \in H$ *such that* $ts = s'h$ *and the action of* t *is given by*

$$t(s \otimes v) = s' \otimes \rho_h v. \tag{2.2}$$

Proof. Straightforward check. □

Lemma 2.2. *Let* $\chi = \chi_\rho$ *and* $\mathrm{Ind}_H^G \chi$ *denote the character of* $\mathrm{Ind}_H^G \rho$. *Then*

$$\mathrm{Ind}_H^G \chi(t) = \sum_{s \in S, s^{-1}ts \in H} \chi\left(s^{-1}ts\right) = \frac{1}{|H|} \sum_{s \in G, s^{-1}ts \in H} \chi\left(s^{-1}ts\right). \tag{2.3}$$

Proof. (2.1) and (2.2) imply

$$\mathrm{Ind}_H^G \chi(t) = \sum_{s \in S, s' = s} \chi(h).$$

Since $s = s'$ implies $h = s^{-1}ts \in H$, we obtain the formula for the induced character. Note also that $\chi(s^{-1}ts)$ does not depend on a choice of s in a left coset. □

Corollary 2.3. *Let H be a normal subgroup in G. Then $\operatorname{Ind}_H^G \chi(t) = 0$ for any $t \notin H$.*

Theorem 2.4. *For any $\rho : G \to \mathrm{GL}(V)$, $\sigma \colon H \to \mathrm{GL}(W)$, we have the identity*

$$\left(\operatorname{Ind}_H^G \chi_\sigma, \chi_\rho\right)_G = \left(\chi_\sigma, \operatorname{Res}_H \chi_\rho\right)_H . \tag{2.4}$$

Here a subindex indicates the group where we take inner product.

Proof. It follows from Frobenius reciprocity, since

$$\dim \operatorname{Hom}_G \left(\operatorname{Ind}_H^G W, V\right) = \dim \operatorname{Hom}_H (W, V) .$$

□

Note that (2.4) can be proved directly from (2.3). Define two maps

$$\operatorname{Res}_H : \mathcal{C}(G) \to \mathcal{C}(H), \ \operatorname{Ind}_H^G : \mathcal{C}(H) \to \mathcal{C}(G),$$

the former is the restriction on a subgroup, the latter is defined by (2.3). Then for any $f \in \mathcal{C}(G), g \in \mathcal{C}(H)$

$$\left(\operatorname{Ind}_H^G g, f\right)_G = \left(g, \operatorname{Res}_H f\right)_H . \tag{2.5}$$

Example 1. Let ρ be a trivial representation of H. Then $\operatorname{Ind}_H^G \rho$ is the permutation representation of G obtained from the natural left action of G on G/H (the set of left cosets).

Example 2. Let $G = S_3$, $H = A_3$, ρ be a non-trivial one dimensional representation of H (one of two possible). Then

$$\operatorname{Ind}_H^G \chi_\rho(1) = 2, \ \operatorname{Ind}_H^G \chi_\rho(12) = 0, \ \operatorname{Ind}_H^G \chi_\rho(123) = -1.$$

Thus, by induction we obtain an irreducible two-dimensional representation of G.

Now consider another subgroup K of $G = S_3$ generated by the transposition (12), and let σ be the (unique) non-trivial one-dimensional representation of K. Then

$$\operatorname{Ind}_K^G \chi_\sigma(1) = 3, \ \operatorname{Ind}_K^G \chi_\sigma(12) = -1, \ \operatorname{Ind}_H^G \chi_\rho(123) = 0.$$

3. Double cosets and restriction to a subgroup

If K and H are subgroups of G one can define the equivalence relation on G: $s \sim t$ iff $s \in KtH$. The equivalence classes are called *double cosets*. We can choose a set of representative $T \subset G$ such that

$$G = \coprod_{s \in T} K \,\mathrm{tH} .$$

We define the set of double cosets by $K\backslash G/H$. One can identify $K\backslash G/H$ with K-orbits on $S = G/H$ in the obvious way and with G-orbits on $G/K \times G/H$ by the formula

$$KtH \to G(K, tH) .$$

Example. Let $\mathbb{F}_q$ be a field of q elements and $G = \mathrm{GL}_2(\mathbb{F}_q) \stackrel{\mathrm{def}}{=} \mathrm{GL}\left(\mathbb{F}_q^2\right)$. Let B be the subgroup of upper-triangular matrices in G. Check that $|G| = (q^2-1)(q^2-q)$, $|B| = (q-1)^2 q$ and therefore $[G:B] = q+1$. Identify G/B with the set of lines $\mathbb{P}^1$ in $\mathbb{F}_q^2$ and $B\backslash G/B$ with G-orbits on $\mathbb{P}^1 \times \mathbb{P}^1$. Check that G has only two orbits on $\mathbb{P}^1 \times \mathbb{P}^1$: the diagonal and its complement. Thus, $|B\backslash G/B| = 2$ and

$$G = B \cup BsB,$$

where

$$s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Theorem 3.1. *Let $T \subset G$ such that $G = \coprod_{s\in T} KtH$. Then*

$$\mathrm{Res}_K \mathrm{Ind}_H^G \rho = \oplus_{s\in T} \mathrm{Ind}_{K\cap sHs^{-1}}^K \rho^s,$$

where

$$\rho_h^s \stackrel{def}{=} \rho_{s^{-1}hs},$$

for any $h \in sHs^{-1}$.

Proof. Let $s \in T$ and $W^s = k(K)(s \otimes V)$. Then by construction, W^s is K-invariant and

$$k(G) \otimes_{k(H)} V = \oplus_{s\in T} W^s.$$

Thus, we need to check that the representation of K in W^s is isomorphic to $\mathrm{Ind}_{K\cap \mathrm{sHs}^{-1}}^K \rho^s$. We define a homomorphism

$$\alpha : \mathrm{Ind}_{K\cap sHs^{-1}}^K V \to W^s$$

by $\alpha(t \otimes v) = ts \otimes v$ for any $t \in K, v \in V$. It is well defined

$$\alpha(th \otimes v - t \otimes \rho_h^s v) = ths \otimes v - ts \otimes \rho_{s^{-1}hs} v = ts\left(s^{-1}hs\right) \otimes v - ts \otimes \rho_{s^{-1}hs} v = 0$$

and obviously surjective. Injectivity can be proved by counting dimensions. $\square$

Example. Let us go back to our example $B \subset \mathrm{SL}_2(\mathbb{F}_q)$. Theorem 3.1 tells us that for any representation ρ of B

$$\mathrm{Ind}_B^G \rho = \rho \oplus \mathrm{Ind}_H^G \rho',$$

where $H = B \cap sBs^{-1}$ is a subgroup of diagonal matrices and

$$\rho' \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \rho \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}$$

Corollary 3.2. *If H is a normal subgroup of G, then*

$$\mathrm{Res}_H \mathrm{Ind}_H^G \rho = \oplus_{s\in G/H} \rho^s.$$

4. Mackey's criterion

To find $\left(\operatorname{Ind}_H^G \chi, \operatorname{Ind}_H^G \chi\right)$ we can use Frobenius reciprocity and Theorem 3.1.

$$\left(\operatorname{Ind}_H^G \chi, \operatorname{Ind}_H^G \chi\right)_G = \left(\operatorname{Res}_H \operatorname{Ind}_H^G \chi, \chi\right)_H = \sum_{s\in T} \left(\operatorname{Ind}_{H\cap sHs^{-1}}^H \chi^s, \chi\right)_H =$$

$$= \sum_{s\in T} (\chi^s, \operatorname{Res}_{H\cap sHs^{-1}} \chi)_{H\cap sHs^{-1}} = (\chi,\chi)_H + \sum_{s\in T\setminus\{1\}} (\chi^s, \operatorname{Res}_{H\cap sHs^{-1}} \chi)_{H\cap sHs^{-1}}.$$

We call two representation *disjoint* if they do not have the same irreducible component, i.e. their characters are orthogonal.

Theorem 4.1. *(Mackey's criterion)* $\operatorname{Ind}_H^G \rho$ *is irreducible iff* ρ *is irreducible and* ρ^s *and* ρ *are disjoint representations of* $H \cap sHs^{-1}$ *for any* $s \in T\setminus\{1\}$.

Proof. Write the condition

$$\left(\operatorname{Ind}_H^G \chi, \operatorname{Ind}_H^G \chi\right)_G = 1$$

and use the above formula. □

Corollary 4.2. *Let* H *be a normal subgroup of* G. *Then* $\operatorname{Ind}_H^G \rho$ *is irreducible iff* ρ^s *is not isomorphic to* ρ *for any* $s \in G/H$, $s \notin H$.

Remark 4.3. Note that if H is normal, then G/H acts on the set of representations of H. In fact, this is a part of the action of the group $\operatorname{Aut} H$ of automorphisms of H on the set of representation of H. Indeed, if $\varphi \in \operatorname{Aut} H$ and $\rho : H \to \mathrm{GL}(V)$ is a representation, then $\rho^\varphi : H \to \mathrm{GL}(V)$ defined by

$$\rho_t^\varphi = \rho_{\varphi(t)},$$

is a new representation of H.

5. Some examples

Let H be a subgroup of G of index 2. Then H is normal and $G = H \cup sH$ for some $s \in G\setminus H$. Suppose that ρ is an irreducible representation of H. There are two possibilities

1. ρ^s is isomorphic to ρ;
2. ρ^s is not isomorphic to ρ.

Hence there are two possibilities for $\operatorname{Ind}_H^G \rho$:

1. $\operatorname{Ind}_H^G \rho = \sigma \oplus \sigma'$, where σ and σ' are two non-isomorphic irreducible representations of G;
2. $\operatorname{Ind}_H^G \rho$ is irreducible.

For instance, let $G = S_5$, $H = A_5$ and $\rho_1, \dots, \rho_5$ be irreducible representation of H, where the numeration is from lecture notes week 3. Then for $i = 1, 2, 3$

$$\operatorname{Ind}_H^G \rho_i = \sigma_i \oplus (\sigma_i \otimes \operatorname{sgn}),$$

here sgn denotes the sign representation. Furthermore, $\operatorname{Ind}_H^G \rho_4 \cong \operatorname{Ind}_H^G \rho_5$ is irreducible. Thus S_5 has two $1, 5, 4$-dimensional irreducible representations and one 6-dimensional.

Now let G be a subgroup of $\mathrm{GL}_2(\mathbb{F}_q)$ of matrices

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

We want to classify irreducible representations of G over $\mathbb{C}$. $|G| = q^2 - q$, G has the following conjugacy classes

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix},$$

in the last case $a \neq 1$. Note that the subgroup H of matrices

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

is normal, $G/H \cong \mathbb{F}_q^* \cong Z_{q-1}$. Therefore G has $q-1$ one-dimensional representations which can be lifted from G/H. That leaves one more representation, its dimension must be $q-1$. We hope to obtain it by induction from H. Let σ be a non-trivial irreducible representation of H (one-dimensional). Then $\dim \operatorname{Ind}_H^G \sigma = q - 1$ as required. Note that for any previously constructed one-dimensional representation ρ of G we have

$$\left(\operatorname{Ind}_H^G \sigma, \rho\right)_G = (\sigma, \operatorname{Res}_H \rho)_H = 0,$$

as $\operatorname{Res}_H \rho$ is trivial. Therefore $\operatorname{Ind}_H^G \sigma$ is irreducible. The character takes values $q-1$, -1 and 0 on the corresponding conjugacy classes.

Remark 5.1. To find all one-dimensional representation of a group G, find its commutator G', which is a subgroup generated by $ghg^{-1}h^{-1}$ for all $g, h \in G$. One-dimensional representations of G are lifted from one-dimensional representations of G/G'.

PROBLEM SET # 4
MATH 252

Due September 30.

1. Let $H \subset K \subset G$. Show that

$$\operatorname{Ind}_K^G \operatorname{Ind}_H^K \rho \cong \operatorname{Ind}_H^G \rho$$

for any representation ρ of H.

2. Let G be the group of matrices

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

where x, y, z are elements of the finite field $\mathbb{F}_5$. Classify irreducible representations of G over $\mathbb{C}$.

Date: September 23, 2005.

REPRESENTATION THEORY
WEEK 5

1. Invariant forms

Recall that a bilinear form on a vector space V is a map

$$B : V \times V \to k$$

satisfying

$$B(cv, dw) = cdB(v, w),\ B(v_1 + v_2, w) = B(v_1, w) + B(v_2, w),\ B(v, w_1 + w_2) = B(v, w_1) + B(v, w_2).$$

One can also think about a bilinear form as a vector in $V^* \otimes V^*$ or as a homomorphism $B : V \to V^*$ given by the formula $B_v(w) = B(v, w)$. A bilinear form is symmetric if $B(v, w) = B(w, v)$ and skew-symmetric if $B(v, w) = -B(w, v)$. Every bilinear form is a sum $B = B^+ + B^-$ of a symmetric and a skew-symmetric form,

$$B^{\pm}(v, w) = \frac{B(v, w) \pm B(w, v)}{2}.$$

Such decomposition corresponds to the decomposition

$$V^* \otimes V^* = S^2V^* \oplus \Lambda^2 V^*. \tag{1.1}$$

A form is *non-degenerate* if $B : V \to V^*$ is an isomorphism, in other words $B(v, V) = 0$ implies $v = 0$.

Let $\rho : G \to \mathrm{GL}(V)$ be a representation. We say that a bilinear form B on V is *G-invariant* if

$$B(\rho_s v, \rho_s w) = B(v, w)$$

for any $v, w \in V$, $s \in G$.

The following properties of an invariant form are easy to check

1. If $W \subset V$ is an invariant subspace, then $W^{\perp} = \{v \in V \mid B(v, W) = 0\}$ is invariant. In particular, $\operatorname{Ker} B$ is invariant.
2. $B : V \to V^*$ is invariant iff $B \in \operatorname{Hom}_G(V, V^*)$.
3. If B is invariant, then B^+ and B^- are invariant.

Lemma 1.1. *Let ρ be an irreducible representation of G, then any bilinear invariant form is non-degenerate. If $\bar{k} = k$, then a bilinear form is unique up to multiplication on a scalar.*

Proof. Follows from (2) and Schur's lemma. □

Date: September 29, 2005.

Corollary 1.2. *A representation ρ of G admits an invariant form iff $\chi_\rho(s) = \chi_\rho(s^{-1})$ for any $s \in G$.*

Lemma 1.3. *If $\bar{k} = k$, then an invariant form on an irreducible representation ρ is either symmetric or skew-symmetric. Let*

$$m_\rho = \frac{1}{|G|} \sum_{s \in G} \chi_\rho\left(s^2\right).$$

Then $m_\rho = 1, 0$ or -1. If $m_\rho = 0$, then ρ does not admit an invariant form. If $m_\rho = \pm 1$, then m_ρ admits a symmetric (skew-symmetric) invariant form.

Proof. Recall that $\rho \otimes \rho = \rho_{\mathrm{alt}} \oplus \rho_{\mathrm{sym}}$.

$$(\chi_{\mathrm{sym}}, 1) = \frac{1}{|G|} \sum_{s \in G} \frac{\chi_\rho(s^2) + \chi_\rho(s^2)}{2},$$

$$(\chi_{\mathrm{alt}}, 1) = \frac{1}{|G|} \sum_{s \in G} \frac{\chi_\rho(s^2) - \chi_\rho(s^2)}{2}.$$

Note that

$$\frac{1}{|G|} \sum_{s \in G} \chi_\rho\left(s^2\right) = (\chi_\rho, \chi_{\rho^*}).$$

Therefore

$$(\chi_{\mathrm{sym}}, 1) = \frac{(\chi_\rho, \chi_{\rho^*}) + m_\rho}{2}, \ (\chi_{\mathrm{alt}}, 1) = \frac{(\chi_\rho, \chi_{\rho^*}) - m_\rho}{2}$$

If ρ does not have an invariant form, then $(\chi_{\mathrm{sym}}, 1) = (\chi_{\mathrm{alt}}, 1) = 0$, and $\chi_{\rho^*} \neq \chi_\rho$, hence $(\chi_\rho, \chi_{\rho^*}) = 0$. Thus, $m_\rho = 0$.

If ρ has a symmetric invariant form, then $(\chi_\rho, \chi_{\rho^*}) = 1$ and $(\chi_{\mathrm{sym}}, 1) = 1$. This implies $m_\rho = 1$. Similarly, if ρ admits a skew-symmetric invariant form, then $m_\rho = -1$. □

Let $k = \mathbb{C}$. An irreducible representation is called *real* if $m_\rho = 1$, *complex* if $m_\rho = 0$ and *quaternionic* if $m_\rho = -1$. Since $\chi_\rho(s^{-1}) = \bar{\chi}_\rho(s)$, then χ_ρ takes only real values for real and quaternionic representations. If ρ is complex then $\chi_\rho(s) \notin \mathbb{R}$ at least for one $s \in G$.

Example. Any irreducible representation of S_4 is real. A non-trivial representation of $\mathbb{Z}_3$ is complex. The two-dimensional representation of quaternionic group is quaternionic.

Exercise. Let $|G|$ be odd. Then any non-trivial irreducible representation of G over $\mathbb{C}$ is complex.

2. Some generalities about field extension

Lemma 2.1. *If* $\operatorname{char} k = 0$ *and* G *is finite, then a representation* $\rho : G \to \mathrm{GL}(V)$ *is irreducible iff* $\operatorname{End}_G(V)$ *is a division ring.*

Proof. In one direction it is Schur's Lemma. In the opposite direction if V is not irreducible, then $V = V_1 \oplus V_2$, then the projectors p_1 and p_2 are intertwiners such that $p_1 \circ p_2 = 0$. □

For any extension F of k and a representation $\rho : G \to \mathrm{GL}(V)$ over k we define by ρ_F the representation $G \to \mathrm{GL}(F \otimes_k V)$.

For any representation $\rho : G \to \mathrm{GL}(V)$ we denote by V^G the subspace of G-invariants in V, i.e.

$$V^G = \{v \in V \mid \rho_s v = v, \forall s \in G\}.$$

Lemma 2.2. $(F \otimes_k V)^G = F \otimes_k V^G$.

Proof. The embedding $F \otimes_k V^G \subset (F \otimes_k V)^G$ is trivial. On the other hand, V^G is the image of the operator

$$p = \frac{1}{|G|} \sum_{s \in G} \tau_s,$$

in particular $\dim V^G$ equals the rank of p. Since rank p does not depend on a field, we have

$$\dim F \otimes_k V^G = \dim (F \otimes_k V)^G.$$

□

Corollary 2.3. *Let* $\rho : G \to \mathrm{GL}(V)$ *and* $\sigma : G \to \mathrm{GL}(W)$ *be two representations over* k*. Then*

$$\operatorname{Hom}_G(F \otimes_k V, F \otimes_k W) = F \otimes \operatorname{Hom}_G(V, W).$$

In particular,

$$\dim_k \operatorname{Hom}_G(V, W) = \dim_F \operatorname{Hom}_G(F \otimes_k V, F \otimes_k W).$$

Proof.

$$\operatorname{Hom}_G(V, W) = (V^* \otimes W)^G.$$

□

Corollary 2.4. *Even if a field is not algebraically closed*

$$dim\ \operatorname{Hom}_G(V, W) = (\chi_\rho, \chi_\sigma).$$

A representation $\rho : G \to \mathrm{GL}(V)$ over k is called *absolutely irreducible* if it remains irreducible after any extension of k. This is equivalent to $(\chi_\rho, \chi_\rho) = 1$. A field is *splitting* for a group G if any irreducible representation is absolutely irreducible. It is not difficult to see that some finite extension of $\mathbb{Q}$ is a splitting field for a finite group G.

3. Representations over $\mathbb{R}$

A bilinear symmetric form B is *positive definite* if $B(v,v) > 0$ for any $v \neq 0$.

Lemma 3.1. *Every representation of a finite group over $\mathbb{R}$ admits positive-definite invariant symmetric form. Two invariant symmetric forms on an irreducible representation are proportional.*

Proof. Let B' be any positive definite form. Define

$$B(v,w) = \frac{1}{|G|}\sum_{s\in G} B'(\rho_s v, \rho_s w).$$

Then B is positive definite and invariant.

Let $Q(v,w)$ be another invariant symmetric form. Then from linear algebra we know that they can be diagonalized in the same basis. Then for some $\lambda \in \mathbb{R}$, $\operatorname{Ker}(Q - \lambda B) \neq 0$. Since $\operatorname{Ker}(Q - \lambda B)$ is invariant, $Q = \lambda B$. □

Theorem 3.2. *Let $\mathbb{R} \subset K$ be a division ring, finite-dimensional over $\mathbb{R}$. Then $\mathbb{R}$ is isomorphic $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ (quaternions).*

Proof. If K is a field, then $K \cong \mathbb{R}$ or $\mathbb{C}$, because $\mathbb{C} = \bar{\mathbb{R}}$ and $[\mathbb{C} : \mathbb{R}] = 2$. Assume that K is not commutative. For any $x \in K \backslash \mathbb{R}$, $\mathbb{R}[x] = \mathbb{C}$. Therefore we have a chain $\mathbb{R} \subset \mathbb{C} \subset K$. Let $f(x) = ixi^{-1}$. Obviously f is an automorphism of K and $f^2 = id$. Hence $K = K^+ \oplus K^-$, where

$$K^{\pm} = \{x \in K \mid f(x) = \pm x\}.$$

Moreover, $K^+K^+ \subset K^+$, $K^-K^- \subset K^+$, $K^+K^- \subset K^-$, $K^-K^+ \subset K^-$. If $x \in K^+$, then $\mathbb{C}[x]$ is a finite extension of $\mathbb{C}$. Therefore $K^+ = \mathbb{C}$. For any nonzero $y \in K^-$ the left multiplication on y defines an isomorphism of K^+ and K^- as vector spaces over $\mathbb{R}$. In particular $\dim_{\mathbb{R}} K^- = \dim_{\mathbb{R}} K^+ = 2$. For any $y \in K^-$, $x \in \mathbb{C}$, we have $y\bar{x} = xy$, therefore $y^2 \in \mathbb{R}$. Moreover, $y^2 < 0$. (If $y^2 > 0$, then $y^2 = b^2$ for some real b and $(y-b)(y+b) = 0$, which is impossible). Put $j = \frac{y}{\sqrt{-y^2}}$. Then we have $k = ij = -ji$, $ki = (ij)i = j$, $K = \mathbb{R}[i,j]$ is isomorphic to $\mathbb{H}$. □

Lemma 3.3. *Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation over $\mathbb{R}$, then there are three possibilities:*

1. *$\mathrm{End}_G(V) = \mathbb{R}$ and $(\chi_\rho, \chi_\rho) = 1$;*
2. *$\mathrm{End}_G(V) \cong \mathbb{C}$ and $(\chi_\rho, \chi_\rho) = 2$;*
3. *$\mathrm{End}_G(V) \cong \mathbb{H}$ and $(\chi_\rho, \chi_\rho) = 4$.*

Proof. Lemma 2.1 and Theorem 3.2 imply that $\mathrm{End}_G(V)$ is isomorphic to $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, $(\chi_\rho, \chi_\rho) = 1, 2$ or 4 as follows from Corollary 2.4.

□

4. Relationship between representations over $\mathbb{R}$ and over $\mathbb{C}$

Hermitian invariant form. Recall that a Hermitian form satisfies the following conditions

$$H(av, bw) = \bar{a}bH(v, w), \; H(w, v) = \bar{H}(v, w).$$

The following Lemma can be proved exactly as Lemma 3.1.

Lemma 4.1. *Every representation of a finite group over $\mathbb{C}$ admits positive-definite invariant Hermitian form. Two invariant Hermitian forms on an irreducible representation are proportional.*

Let $\rho : G \to \mathrm{GL}(V)$ be a representation over $\mathbb{C}$. Denote by $V^{\mathbb{R}}$ a vector space V as a vector space over $\mathbb{R}$ of double dimension. Denote by $\rho^{\mathbb{R}}$ the representation of G in $V^{\mathbb{R}}$. Check that

$$\chi_{\rho^{\mathbb{R}}} = \chi_\rho + \bar{\chi}_\rho. \tag{4.1}$$

Theorem 4.2. *Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation over $\mathbb{C}$.*

1. *If ρ can be realized by matrices with real entries, then ρ admits an invariant symmetric form.*
2. *If $\mathrm{End}_G\left(V^{\mathbb{R}}\right) = \mathbb{C}$, then ρ is complex, i.e. ρ does not admit a bilinear invariant symmetric form.*
3. *If $\mathrm{End}_G\left(V^{\mathbb{R}}\right) = \mathbb{H}$, then ρ admits an invariant skew-symmetric form.*

Proof. (1) follows from Lemma 3.1. For (2) use (4.1). Since $(\chi_\rho, \chi_\rho) = 2$ by Lemma 3.3, then $\chi_\rho \neq \bar{\chi}_\rho$, and therefore ρ is complex.

Finally let us prove (3). Let $j \in \mathrm{End}_G\left(V^{\mathbb{R}}\right) = \mathbb{H}$, then $j(bv) = \bar{b}v$ for any $b \in \mathbb{C}$. Let H be a positive-definite Hermitian form on V. Then

$$Q(v, w) = H(jw, jv)$$

is another invariant positive-definite Hermitian form. By Lemma 4.1 $Q = \lambda H$ for some $\lambda > 0$. Since $j^2 = -1$, $\lambda^2 = 1$ and therefore $\lambda = 1$. Thus,

$$H(v, w) = H(jw, jv).$$

Set

$$B(v, w) = H(jv, w).$$

Then B is a bilinear invariant form, and

$$B(w, v) = H(jw, v) = H\left(jv, j^2 w\right) = -H(jv, w) = -B(v, w),$$

hence B is skew-symmetric. ∎

Corollary 4.3. *Let σ be an irreducible representation of G over $\mathbb{R}$. There are three possibilities for σ*

σ is absolutely irreducible and $\chi_\sigma = \chi_\rho$ for some real representation ρ of G over $\mathbb{C}$;
$\chi_\sigma = \chi_\rho + \bar{\chi}_\rho$ for some complex representation ρ of G over $\mathbb{C}$;
$\chi_\sigma = 2\chi_\rho$ for some quaternionic representation ρ of G over $\mathbb{C}$.

5. Representations of symmetric group

Let $\mathcal{A}$ denote the group algebra $\mathbb{Q}(S_n)$. We will see that $\mathbb{Q}$ is a splitting field for S_n. We realize irreducible representation of S_n as minimal left ideals in $\mathcal{A}$.

Conjugacy classes are enumerated by partitions $m_1 \geq \dots \geq m_k > 0$, $m_1 + \dots + m_k = n$. To each partition we associate the table of n boxes with rows of length $m_1, \dots, m_k$, it is called a *Young diagram. Young tableau* is a Young diagram with entries $1, \dots, n$ in boxes. Given a Young tableau λ, we denote by P_λ the subgroup of permutations preserving rows and by Q_λ the subgroup of permutations preserving columns. Introduce the following elements in $\mathcal{A}$

$$a_\lambda = \sum_{p \in P_\lambda} p,\ b_\lambda = \sum_{q \in Q_\lambda} (-1)^q q,\ c_\lambda = a_\lambda b_\lambda.$$

The element c_λ is called *Young symmetrizer.*

Theorem 5.1. *$V_\lambda = \mathcal{A}c_\lambda$ is a minimal left ideal in $\mathcal{A}$, therefore V_λ is irreducible. V_λ is isomorphic to V_μ iff the Young tableaux μ and λ have the same Young diagram. Any irreducible representation of S_n is isomorphic to V_λ for some Young tableau λ.*

Note that the last assertion of Theorem follows from the first two, since the number of Young diagrams equals the number of conjugacy classes.

Examples. For partition (n), $c_\lambda = a_\lambda = \sum_{s \in S_n} s$, and the representation is trivial. For (1,...,1), $c_\lambda = b_\lambda = \sum_{s \in S_n} (-1)^s s$.

Let us consider partition $(n-1, 1)$. Then

$$c_\lambda = \left(\sum_{s \in S_{n-1}} s \right) (1 - (1n)).$$

Clearly, $a_\lambda c_\lambda = c_\lambda$, therefore $\operatorname{Res}_{S_{n-1}} V_\lambda$ contains the trivial representation. Let

$$V = \operatorname{Ind}_{S_{n-1}}^{S_n} (\text{triv}).$$

Note that V is the permutation representation of S_n. By Frobenius reciprocity we have a homomorphism $V \to V_\lambda$. Therefore $V = V_\lambda \oplus \text{triv}$.

Now we will prove Theorem 5.1. First, note that S_n acts on the Young tableaux of the same shape, and

$$a_{s(\lambda)} = s a_\lambda s^{-1},\ b_{s(\lambda)} = s b_\lambda s^{-1},\ c_{s(\lambda)} = s c_\lambda s^{-1}.$$

Check yourself the following

Lemma 5.2. *If $s \in S_n$, but $s \notin P_\lambda Q_\lambda$, then there exists two numbers i and j in the same row of λ and in the same column of $s(\lambda)$.*

It is clear also that for any $p \in P_\lambda$, $q \in Q_\lambda$

$$p a_\lambda = a_\lambda p = a_\lambda,\ q b_\lambda = b_\lambda q = (-1)^q b_\lambda,\ p c_\lambda q = (-1)^q c_\lambda.$$

Lemma 5.3. *Let $y \in \mathcal{A}$ such that for any $p \in P_\lambda$, $q \in Q_\lambda$*

$$pyq = (-1)^q \, y.$$

Then $y \in \mathbb{Q}c_\lambda$.

Proof. It is clear that y has a form

$$\sum_{s \in P_\lambda \backslash S_n / Q_\lambda} d_s \sum_{p \in P_\lambda, q \in Q_\lambda} (-1)^q \, psq = \sum_{s \in P_\lambda \backslash S_n / Q_\lambda} d_s a_\lambda s b_\lambda,$$

for some $d_s \in \mathbb{Q}$. We have to show that if $s \notin P_\lambda Q_\lambda$ then $a_\lambda s b_\lambda = 0$. That follows from Lemma 5.2. There exists $(ij) \in P_\lambda \cap Q_{s(\lambda)}$. Then

$$a_\lambda s b_\lambda s^{-1} = a_\lambda b_{s(\lambda)} = a_\lambda \,(ij)\,(ij)\, b_{s(\lambda)} = a_\lambda b_{s(\lambda)} = -a_\lambda b_{s(\lambda)} = 0.$$

□

Corollary 5.4. $c_\lambda \mathcal{A} c_\lambda \subset \mathbb{Q}c_\lambda$.

Lemma 5.5. *Let W be a left ideal in a group algebra $k\,(G)$ ($\operatorname{char} k = 0$). Then $W^2 = 0$ implies $W = 0$.*

Proof. Since $k\,(G)$ is completely reducible $k\,(G) = W \oplus W'$, where W' is another left ideal. Let $y \in \operatorname{End}_G(k\,(G))$ such that $y_{|W} = \mathrm{Id}$, $y\,(W') = 0$. But we proved that any $y \in \operatorname{End}_G(k\,(G))$ is a right multiplication on some $u \in k\,(G)$ (see lecture notes 3). Then we have $u^2 = u$, $W = \mathcal{A}u$, in particular $u \in W$. If $W \neq 0$, then $u \neq 0$ and $u^2 = u \neq 0$. Hence $W^2 \neq 0$. □

Corollary 5.6. *$\mathcal{A}c_\lambda$ is a minimal left ideal.*

Proof. Let $W \subset \mathcal{A}c_\lambda$ be a left ideal. Then either $c_\lambda W = \mathbb{Q}c_\lambda$ or $c_\lambda W = 0$ by Corollary 5.4. In the former case $W = \mathcal{A}c_\lambda W = \mathcal{A}c_\lambda$. In the latter case $W^2 \subset \mathcal{A}c_\lambda W = 0$, and $W = 0$ by Lemma 5.5.

□

Corollary 5.7. $c_\lambda^2 = n_\lambda c_\lambda$, *where* $n_\lambda = \frac{n!}{\dim V_\lambda}$.

Proof. From the proof of Lemma 5.5, $c_\lambda = n_\lambda u$ for some idempotent $u \in \mathbb{Q}\,(S_n)$. Therefore $c_\lambda = n_\lambda u$. To find n_λ note that $\operatorname{tr}_{k(G)} u = \dim V_\lambda$, $\operatorname{tr}_{k(G)} c_\lambda = |S_n| = n!$. □

Lemma 5.8. *Order partitions lexicographically. If $\lambda > \mu$, then there exists i, j in the same row of λ and in the same column of μ.*

Proof. Check yourself. □

Corollary 5.9. *If $\lambda < \mu$, then $c_\lambda \mathcal{A} c_\mu = 0$.*

Proof. Sufficient to check that $c_\lambda s c_\mu = 0$ for any $s \in S_n$, which is equivalent to

$$c_\lambda s c_\mu s^{-1} = c_\lambda c_{s(\mu)} = 0.$$

Let $(ij) \in Q_\lambda \cap P_{s(\mu)}$. Then

$$c_\lambda (ij) (ij) c_{s(\mu)} = c_\lambda c_{s(\mu)} = -c_\lambda c_{s(\mu)} = 0.$$

□

Lemma 5.10. *V_λ and V_μ are isomorphic iff λ and μ have the same Young diagram.*

Proof. If λ and μ have the same diagram, then $\lambda = s(\mu)$ for some $s \in S_n$ and $\mathcal{A}c_\lambda = \mathcal{A}sc_\mu s^{-1} = \mathcal{A}c_\mu s^{-1}$. Assume $\lambda > \mu$, then $c_\lambda \mathcal{A} c_\mu = 0$ and $c_\lambda \mathcal{A} c_\lambda \neq 0$. Therefore $\mathcal{A}c_\lambda$ and $\mathcal{A}c_\mu$ are not isomorphic. □

Corollary 5.11. *If λ and μ have different diagrams, then $c_\lambda \mathcal{A} c_\mu = 0$.*

Proof. If $c_\lambda \mathcal{A} c_\mu \neq 0$, then $\mathcal{A}c_\lambda \mathcal{A} c_\mu = \mathcal{A}c_\mu$. On the other hand $\mathcal{A}c_\lambda \mathcal{A}$ has only components isomorphic to V_λ. Contradiction. □

Lemma 5.12. *Let $\rho : S_n \to \mathrm{GL}(V)$ be an arbitrary representation. Then the multiplicity of V_λ in V equals the rank of $\rho(c_\lambda)$.*

Proof. The rank of c_λ is 1 in V_λ and 0 in any V_μ with another Young diagram. □

PROBLEM SET # 5
MATH 252

Due October 7.

1. Classify irreducible representations of A_4 (even permutations) over $\mathbb{R}$ and over $\mathbb{C}$. What is the splitting field for A_4.

2. Let G be a finite group, r be the number of conjugacy classes in G and s be the number of conjugacy classes in G preserved by the involution $g \to g^{-1}$. Prove that the number of irreducible representations of G over $\mathbb{R}$ is equal to $\frac{r+s}{2}$.

3. If λ is a Young tableau, then the conjugate tableau λ' is obtained from λ by symmetry about diagonal (rows and columns switch). Show that $V_{\lambda'}$ is isomorphic $V_\lambda \otimes \text{sgn}$, where sgn is one-dimensional sign representation. (Hint: you probably have to show that $\mathbb{Q}(S_n)\, a_\lambda b_\lambda$ and $\mathbb{Q}(S_n)\, b_\lambda a_\lambda$ are isomorphic).

Date: September 29, 2005.

REPRESENTATION THEORY
WEEK 6

1. Dimension formulae

For a Young tableau (diagram) λ, λ_i denotes the number of boxes in the i-th row. We write $\alpha \in \lambda$ if α is a box of λ. Let $|\lambda|$ denote the number of boxes in λ and $\lambda - \alpha$ denote a diagram which can be obtained from λ by removing one box. For example, for partition $\lambda = (5, 3, 1)$ the possible $\lambda - \alpha$ are (4,3,1), (5,2,1) and (5,3).

Theorem 1.1. $\mathrm{Res}_{S_{n-1}} V_\lambda = \oplus V_{\lambda-\alpha}$.

Lemma 1.2. *Let $|\mu| = |\lambda| - 1$. If the diagram of μ is different from the diagram of $\lambda - \alpha$ for all possible α, then there are i, j either in the same row of λ and in the same column of μ or in the same row of μ and in the same column of λ.*

Proof. Let λ and μ do not satisfy the condition of Lemma. Choose the smallest k such that $\lambda_k \neq \mu_k$. Assume first, that $\mu_k > \lambda_k$, then by pigeon hole principle one can find two entries of k-th row of μ in the same column of λ. (We assume that this does not happen with the first $k-1$ rows).

Assume now that $\lambda_k > \mu_k$. Since λ has just one more entry than μ, $\lambda_k > \mu_k + 1$ implies that two entries in the k-th row of λ are in the same column of μ. Therefore $\lambda_k = \mu_k + 1$, moreover the last entry appears in the first k rows of λ. In this case we move to the next row, and step by step prove that $\lambda_i = \mu_i$ for all $i \neq k$. Hence $\mu = \lambda - \alpha$. □

Lemma 1.3. *If the diagram of μ is different from the diagram of $\lambda - \alpha$ for all possible α, then $c_\mu \mathcal{A} c_\lambda = 0$.*

Proof. Let $s \in S_n$. First assume that there are two entries in the same row of μ and in the same column of λ. Then the same is true for μ and $s(\lambda)$ for any $s \in S_n$ as we can see from the proof of Lemma 1.2. Hence for this pair of entries i, j we have

$$a_\mu (ij)^2 b_{s(\lambda)} = a_\mu b_{s(\lambda)} = -a_\mu b_{s(\lambda)} = 0.$$

Therefore $a_\mu s b_\lambda s^{-1} = 0$ and $a_\mu s b_\lambda = 0$ for any $s \in S_n$. That implies $a_\mu \mathcal{A} b_\lambda = 0$. Similarly we can prove that if there are two entries in the same column of μ and in the same row of λ, then $b_\mu \mathcal{A} a_\lambda = 0$. Together that implies $c_\mu \mathcal{A} c_\lambda = 0$. □

Corollary 1.4. *If a Young diagram μ can not be obtained from λ by removing one box, then the multiplicity of V_μ in $\mathrm{Res}_{S_{n-1}} V_\lambda$ is zero.*

Date: October 7, 2005.

Proof. Follows from Lemma 5.12 (lecture notes 5). □

Lemma 1.5. *Let $\mu = \lambda - \alpha$. Then $c_\mu c_\lambda \neq 0$. Therefore $f : V_\mu \to V_\lambda$ given by $f(x) = xc_\lambda$ is an injective homomorphism of S_{n-1}-modules.*

Proof. If we write

$$c_\mu c_\lambda = \sum_{s\in S_n} u_s s,\ c_\mu^2 = \sum_{s\in S_{n-1}} v_s s,$$

then as one can easily see that $u_s = v_s$ for any $s \in S_{n-1}$. □

Lemma 1.6. *If $\mu = \lambda - \alpha$, then $c_\mu \mathcal{A} c_\lambda \subset \mathbb{Q} c_\mu c_\lambda$.*

Proof. First, prove that if there is no $(ij) \in Q_\mu P_{s(\lambda)}$ then $s \in Q_\lambda P_\lambda$. Thus, $b_\mu s a_\mu \neq 0$, or equivalently $b_\mu a_{s(\lambda)} \neq 0$, implies $s \in Q_\lambda P_\lambda$. Then prove that for any $s \in Q_\lambda P_\lambda$, $c_\mu s c_\lambda \in \mathbb{Q} c_\mu c_\lambda$. □

Corollary 1.4, Lemma 1.5 and Lemma 1.6 imply Theorem 1.1.

Corollary 1.7.

$$\dim V_\lambda = \sum \dim V_{\lambda-\alpha}.$$

Remark 1.8. Every function $f(\lambda)$ on the set of Young diagrams satisfying

$$f(\lambda) = \sum f(\lambda - \alpha),\ f(1) = 1 \tag{1.1}$$

coincides with $\dim V_\lambda$.

Corollary 1.9. $\operatorname{Ind}_{S_n}^{S_{n+1}} V_\lambda = \oplus V_\mu$, *where μ runs the set of all diagrams obtained from λ by adding one box.*

A Young tableau is *standard* if entries in every row and entries in every column are in increasing order.

Corollary 1.10. $\dim V_\lambda$ *equals the number of all standard tableaux on a diagram λ.*

Proof. Check that the number d_λ of standard tableaux satisfies (1.1). □

For a box $\alpha \in \lambda$, let h_α be the hook diagram containing α, all boxes below α and all boxes to the right of α. Let

$$h(\lambda) = \prod_{\alpha\in\lambda} |h_\alpha|.$$

Example. If λ is (3,2,1), then $h(\lambda) = 45$.

Lemma 1.11. *Let $\lambda = (\lambda_1, \dots, \lambda_k)$ and $\bar\lambda = (\lambda_1 + k - 1, \lambda_2 + k - 2, \dots, \lambda_k)$. Then*

$$h(\lambda) = \frac{\bar\lambda_1! \dots \bar\lambda_k!}{\prod_{i<j} (\bar\lambda_i - \bar\lambda_j)}. \tag{1.2}$$

Proof. Direct calculation. ∎

Lemma 1.12. *Let*

$$V(x_1, \dots, x_k) = \prod_{i<j} (x_i - x_j).$$

Then

$$\sum_{i=1}^{k} x_i \left(V(x_1, \dots, x_k) - V(x_1, \dots, x_i - 1, \dots, x_k)\right) = \frac{k(k-1)}{2} V(x_1, \dots, x_k). \tag{1.3}$$

Proof. Since V is a skew symmetric polynomial of x, it is easy to check that $(x_i - x_j)$ divides the left hand side of the identity. Since the degree of the left hand side polynomial is $\frac{k(k-1)}{2}$, the same as the degree of V, the LHS polynomial is proportional to V. The leading coefficient of LHS is the same as of

$$\sum_{i=1}^{k} x_i \frac{\partial}{\partial x_i} V(x_1, \dots, x_k) = \frac{k(k-1)}{2} V(x_1, \dots, x_k).$$

That proves the identity. ∎

Lemma 1.13.

$$\frac{n}{h(\lambda)} = \sum \frac{1}{h(\lambda - \alpha)}.$$

Proof. Using (1.2) write

$$n \frac{V\left(\bar{\lambda}_1, \dots, \bar{\lambda}_k\right)}{\bar{\lambda}_1! \dots \bar{\lambda}_k!} = \sum_{i=1}^{k} \frac{V\left(\bar{\lambda}_1, \dots, \bar{\lambda}_i - 1, \dots, \bar{\lambda}_k\right)}{\bar{\lambda}_1! \dots \left(\bar{\lambda}_i - 1\right)! \dots \lambda_k!}.$$

This is equivalent to

$$nV\left(\bar{\lambda}_1, \dots, \bar{\lambda}_k\right) = \sum_{i=1}^{k} \bar{\lambda}_i V\left(\bar{\lambda}_1, \dots, \bar{\lambda}_i - 1, \dots, \bar{\lambda}_k\right). \tag{1.4}$$

Use now that $\bar{\lambda}_1 + \dots + \bar{\lambda}_k = n + \frac{k(k-1)}{2}$ and apply (1.3) to prove (1.4). ∎

Corollary 1.14. *(Hook formula)* $\dim V_\lambda = \frac{|\lambda|!}{h(\lambda)}$.

Proof. Just check that $\frac{|\lambda|!}{h(\lambda)}$ satisfies (1.1). ∎

2. Representations of GL_k.

Matrix coefficients. Let $\rho : G \to \mathrm{GL}(V)$ be a (finite-dimensional) representation. For any $\varphi \in V^*, v \in V$ define

$$f_{v,\varphi}(s) = \langle \varphi, \rho_s v \rangle .$$

This function f is called a *matrix coefficient*.

Let $G = \mathrm{GL}_k = \mathrm{GL}(\mathbb{C}^k)$ and $\mathbb{C}[G]$ denote the space of all polynomial functions on G. We call a representation $\rho : G \to \mathrm{GL}(V)$ *polynomial* if $f_{v,\varphi} \in \mathbb{C}[G]$ for all $v \in V, \varphi \in V^*$.

Examples. The standard representation in the space $E = \mathbb{C}^k$ is polynomial, but the dual representation in E^* is not.

The whole space $\mathbb{C}[G]$ has a natural structure of a representation if we put

$$R_g f(x) = f(xg).$$

Check that the space $\mathbb{C}_n[G]$ of homogeneous polynomials of degree n is invariant. Thus, there is a decomposition

$$\mathbb{C}[G] = \oplus_{n=0}^{\infty} \mathbb{C}_n[G].$$

Let $\rho : G \to \mathrm{GL}(V)$ be a polynomial representation. For any $\varphi \in V^*$ define a map $\rho'_\varphi : V \to \mathbb{C}[G]$ by the formula

$$\rho'_\varphi(v) = f_{v,\varphi}.$$

Check that this map is an intertwiner, i.e.

$$\rho'_\varphi(\rho_g v) = R_g \rho'_\varphi(v).$$

That implies

Lemma 2.1. *Every irreducible polynomial representation of G is a subrepresentation in $\mathbb{C}[G]$.*

Lemma 2.2. *Consider the representation of G in $(E^*)^{\otimes n} \otimes E^{\otimes n}$ given by the formula*

$$\rho_g(\varphi_1 \otimes \cdots \otimes \varphi_n \otimes v_1 \otimes \cdots \otimes v_n) = \varphi_1 \otimes \cdots \otimes \varphi_n \otimes gv_1 \otimes \cdots \otimes gv_n,$$

for all $\varphi_i \in E^*, v_j \in E$.

The map $\pi : (E^*)^{\otimes n} \otimes E^{\otimes n} \to \mathbb{C}_n[G]$ given by

$$\varphi \otimes v \mapsto f_{v,\varphi}$$

for each $\varphi \in (E^*)^{\otimes n}$, $v \in E^{\otimes n}$, is surjective.

Proof. Let $e_1, \dots, e_k$ be a basis in E and $f_1, \dots, f_k$ be the dual basis in E^*. Then

$$f_{j_1} \otimes \cdots \otimes f_{j_n} \otimes e_{i_1} \otimes \cdots \otimes e_{i_n} \mapsto g_{i_1 j_1} \dots g_{i_n j_n},$$

where g_{ij} is a matrix entry of a matrix g in the basis $e_1, \dots, e_k$. Thus, the monomial basis of $\mathbb{C}_n[G]$ belongs to the image of π. ∎

Remark 2.3. Here I made a mistake during the lecture. To get an isomorphism we have to consider $S^n(E^* \otimes E) \subset (E^*)^{\otimes n} \otimes E^{\otimes n}$.

To classify polynomial irreducible representation of G we have to find all irreducible subrepresentations of $E^{\otimes n}$. We will do this in the next section.

3. Duality between GL_k and S_n

Consider the representation $\rho : S_n \to \mathrm{GL}(E^{\otimes n})$ defined by the formula

$$\rho_s(v_1 \otimes \cdots \otimes v_n) = v_{s(1)} \otimes \cdots \otimes v_{s(n)},$$

and the representation $\rho : \mathrm{GL}_k \to \mathrm{GL}(E^{\otimes n})$ defined by

$$\rho_g(v_1 \otimes \cdots \otimes v_n) = gv_1 \otimes \cdots \otimes gv_n.$$

We see immediately that $\rho_s \circ \rho_g = \rho_g \circ \rho_s$ for any $s \in S_n$, $g \in \mathrm{GL}_k$. Thus we can consider ρ as the representation of the direct product $S_n \times \mathrm{GL}_k$.

Theorem 3.1. *Let $\Gamma_{n,k}$ denote the set of all Young diagrams with n boxes such that the number of rows is not bigger than k. Then*

$$E^{\otimes n} = \oplus_{\lambda \in \Gamma_{n,k}} V_\lambda \boxtimes W_\lambda,$$

where V_λ is the irreducible representation of S_n associated with λ and W_λ is an irreducible representation of GL_k. Moreover, W_λ and W_μ are not isomorphic if $\lambda \neq \mu$.

Corollary 3.2. *Fill the boxes of λ in some way. Then $\rho_{c_\lambda}(E^{\otimes n})$ is an invariant subspace isomorphic to W_λ.*

Example. Let $\lambda = (n)$ be a one row diagram. Then V_λ is the trivial representation of S_n, $c_\lambda = \sum_{s \in S_n} s$, and $W_\lambda = S^n(E)$.

If $k > n$, and $\lambda = (1, \dots, 1)$ (one row), then V_λ is the sign representation, $c_\lambda = \sum_{s \in S_n} (-1)^s s$, and $W_\lambda = \Lambda^n(E)$.

To prove Theorem 3.1 we need the following general statement.

Theorem 3.3. *Let $\rho : G \to \mathrm{GL}(V)$, $\sigma \colon K \to \mathrm{GL}(V)$ be two representations in the same vector space V over algebraically closed F. Let*

$$\mathrm{End}_G(V) = \sigma(F(K))$$

and ρ is completely reducible. Then

$$V = \oplus_{i=1}^m V_i \boxtimes W_i,$$

where V_i is an irreducible representation of G, W_i is an irreducible representation of K. Moreover, V_i is not isomorphic to V_j if $i \neq j$ and similarly, W_i is not isomorphic to W_j if $i \neq j$.

Proof. Since ρ is completely reducible, one can write

$$V = \oplus_{i=1}^{m} (V_i \otimes W_i),$$

the action of G is trivial on W_i. Then

$$\mathrm{End}_G(V) = \prod_{i=1}^{m} \mathrm{End}_F(W_i).$$

Thus, $\sigma\colon F(K) \to \mathrm{End}_F(W_i)$ is surjective, that implies that each W_i is irreducible over K and $W_i \not\cong W_j$ if $i \neq j$. ☐

Remark 3.4. In general, we say that G and K satisfying the conditions of Theorem 3.3 form a *dual pair*. Such situation often happens in representation theory. The simplest example is an action of $G \times G$ in $k(G)$ given by

$$R_{(g,h)} \sum_{s \in G} u_s s = \sum_{s \in G} u_s g s h^{-1}.$$

Lemma 3.5. *In the situation of Theorem 3.1 we have*

$$\mathrm{End}_{S_n}\left(E^{\otimes n}\right) = \rho\left(\mathbb{C}\left(\mathrm{GL}_k\right)\right).$$

Proof. Let M_k denote the algebra $\mathrm{End}_{\mathbb{C}}(E)$, in other words, M_k is the matrix algebra. First,

$$\mathrm{End}_{\mathbb{C}}\left(E^{\otimes n}\right) = M_k^{\otimes n}.$$

Thus, we are looking at the S_n invariant subalgebra

$$\mathrm{End}_{S_n}\left(E^{\otimes n}\right) = \left(M_k^{\otimes n}\right)^{S_n} = S^n(M_k).$$

In other words, $\mathrm{End}_{\mathbb{C}}(E^{\otimes n})$ is spanned by

$$\sum_{s \in S_n} m_{s(1)} \otimes \cdots \otimes m_{s(n)}$$

for all possible $m_1, \dots, m_n \in M_k$.

Our next claim is that $\mathrm{End}_{\mathbb{C}}(E^{\otimes n})$ is the span of $m \otimes \cdots \otimes m$ for all possible $m \in M_k$. It follows from the following

Lemma 3.6. *For an arbitrary vector space V, $S^n(V)$ is spanned by v^n for all $v \in V$.*

Proof. Everybody knows the formula

$$4xy = (x+y)^2 - (x-y)^2,$$

which proves the statement for $n = 2$. Less know is the following general formula

$$2^n x_1 \dots x_n = \sum_{i_2=0,1,\dots,i_n=0,1} (-1)^{i_2+\dots+i_n} \left(x_1 + (-1)^{i_2} x_2 + \dots + (-1)^{i_n} x_n\right)^n.$$

☐

Now let U be the span of $g \otimes \cdots \otimes g \, (g \in \mathrm{GL}_k)$ in $\mathrm{End}_{\mathbb{C}}\left(E^{\otimes n}\right) = S^n\left(M_k\right)$. By definition, $U = \rho\left(\mathbb{C}\left(\mathrm{GL}_k\right)\right)$. Note that GL_k is a dense subset in M_k, therefore U is a dense subset in $S^n\left(M_k\right)$. But U is a linear subspace in $S^n\left(M_k\right)$. Hence $U = S^n\left(M_k\right) = \mathrm{End}_{\mathbb{C}}\left(E^{\otimes n}\right)$. □

Note that Lemma 3.5 together with Theorem 3.3 imply that

$$E^{\otimes n} = \oplus_{\lambda \in \Gamma'} V_\lambda \boxtimes W_\lambda,$$

for some set Γ' of Young diagrams with n boxes. It is left to show that $\Gamma' = \Gamma_{n,k}$. Obviously, Γ' consists of all diagrams λ for which $W_\lambda = c_\lambda\left(E^{\otimes n}\right) \neq 0$. Fill the boxes of λ in increasing order from 1 to n from left to right starting from the top and consider c_λ defined by this tableaux. An element $v = e_{i_1} \otimes \cdots \otimes e_{i_n}$ of the standard basis in $E^{\otimes n}$ can be represented by the same tableau with entries $e_{i_1}, \dots, e_{i_n}$. If λ has more than k rows, then one can find e_j which appears twice in the same column. Then $b_\lambda(v) = 0$, and therefore $c_\lambda v = 0$. Since this holds for any basis vector, we have $c_\lambda\left(E^{\otimes n}\right) = 0$. Hence $\lambda \notin \Gamma'$ if λ has more than k rows. On the other hand, if λ has k or less rows, one can check that

$$c_\lambda\left(e_1^{\otimes \lambda_1} \otimes \cdots \otimes e_k^{\otimes \lambda_k}\right) \neq 0.$$

Therefore $\Gamma' = \Gamma_{n,k}$. Theorem 3.1 is proven.

PROBLEM SET # 6
MATH 252

Due October 14.

Use notations of lecture notes 5 and 6.

1. Let $\lambda = (n-k, 1, ..., 1)$ be the hook diagram with n boxes. Show that V_λ is isomorphic to $\Lambda^k V$, where V is $n-1$-dimensional subrepresentation in the standard permutation representation of S_n.(Hint: Use Lemma 5.12 in Lecture notes 5)

2. Show that the GL_k-representation $W_\lambda \otimes E$ decomposes into direct sum of W_μ for all $\mu \in \Gamma_{n+1,k}$ which can be obtained from λ by adding one box. (Hint: check when $\rho_{c_\mu}\rho_{c_\lambda}(E^{\otimes n+1}) \neq 0$.)

Date: October 6, 2005.

REPRESENTATION THEORY
WEEK 7

1. Characters of GL_k and S_n

A character of an irreducible representation of GL_k is a polynomial function constant on every conjugacy class. Since the set of diagonalizable matrices is dense in GL_k, a character is defined by its values on the subgroup of diagonal matrices in GL_k. Thus, one can consider a character as a polynomial function of $x_1, \dots, x_k$. Moreover, a character is a symmetric polynomial of $x_1, \dots, x_k$ as the matrices $\operatorname{diag}(x_1, \dots, x_k)$ and $\operatorname{diag}\left(x_{s(1)}, \dots, x_{s(k)}\right)$ are conjugate for any $s \in S_k$.

For example, the character of the standard representation in E is equal to $x_1 + \dots + x_k$ and the character of $E^{\otimes n}$ is equal to $(x_1 + \dots + x_k)^n$.

Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq 0$. Let D_λ denote the determinant of the $k \times k$-matrix whose i, j entry equals $x_i^{\lambda_j}$. It is clear that D_λ is a skew-symmetric polynomial of $x_1, \dots, x_k$. If $\rho = (k-1, \dots, 1, 0)$ then $D_\rho = \prod_{i \leq j} (x_i - x_j)$ is the well known Vandermonde determinant. Let

$$S_\lambda = \frac{D_{\lambda+\rho}}{D_\rho}.$$

It is easy to see that S_λ is a symmetric polynomial of $x_1, \dots, x_k$. It is called a *Schur polynomial*. The leading monomial of S_λ is the $x^{\lambda_1} \dots x_k^{\lambda_k}$ (if one orders monomials lexicographically) and therefore it is not hard to show that S_λ form a basis in the ring of symmetric polynomials of $x_1, \dots, x_k$.

Theorem 1.1. *The character of W_λ equals to S_λ.*

I do not include a proof of this Theorem since it uses beautiful but hard combinatoric. The proof is much easier in general framework of Lie groups and is included in 261A course.

Exercise. Check that

$$\dim W_\lambda = \prod_{i<j} \frac{\left(\bar{\lambda}_i - \bar{\lambda}_j\right)}{(\rho_i - \rho_j)} = \frac{\prod_{i<j} \left(\bar{\lambda}_i - \bar{\lambda}_j\right)}{(k-1)!\,(k-2)!\dots 1!},$$

if $\bar{\lambda} = \lambda + \rho$.

Now we use Schur-Weyl duality to establish the relation between characters of S_n and GL_k. Recall that the conjugacy classes in S_n are given by partitions of n. Let $C(\mu)$ be the class associated with the partition μ in the natural way. Let ρ denote

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the representation of $S_n \times \mathrm{GL}_k$ in $E^{\otimes n}$. Let r be the number of rows in μ. Then one can see that

$$\operatorname{tr}(\rho_{s\times g}) = (x_1^{\mu_1} + \dots + x_k^{\mu_1}) \dots (x_1^{\mu_r} + \dots + x_k^{\mu_r}), \tag{1.1}$$

for any $s \in C(\mu)$ and a diagonal $g \in \mathrm{GL}_k$. Denote by P_μ the polynomial in the right hand side of the identity. Let χ_λ be the character of V_λ. Since

$$\operatorname{tr}(\rho_{s\times g}) = \sum_{\lambda\in\Gamma_{n,k}} \chi_\lambda(s) S_\lambda(g),$$

one obtains the following remarkable relation

$$P_\mu = \sum_{\lambda\in\Gamma_{n,k}} \chi_\lambda(s) S_\lambda. \tag{1.2}$$

2. Representations of compact groups

Let G be a group and a topological space. We say that G is a *topological group* if the multiplication map $G \times G \to G$ and the inverse $G \to G$ are continuous maps. Naturally, G is compact if it is compact topological space.

Examples. The circle

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}.$$

A torus $T^n = S^1 \times \dots \times S^1$.

Unitary group

$$U_n = \left\{X \in \mathrm{GL}_n \mid \bar{X}^t X = 1_n\right\}.$$

Special unitary group

$$SU_n = \{X \in U_n \mid \det X = 1\}.$$

Orthogonal group

$$O_n = \left\{x \in \mathrm{GL}_n(\mathbb{R}) \mid X^t X = 1_n\right\}.$$

Special orthogonal group

$$SO_n = \{X \in O_n \mid \det X = 1\}.$$

Theorem 2.1. *Let G be a compact group. There exists a unique measure on G such that*

$$\int_G f(ts)\,dt = \int_G f(t)\,dt,$$

for any integrable function f on G and any $s \in G$, and $\int_G dt = 1$.

In the same way there exists a measure $d't$ such that

$$\int_G f(st)\,dt = \int_G f(t)\,d't, \int_G d't = 1.$$

Moreover, for a compact group $dt = d't$.

The measure dt $(d't)$ is called right-invariant (left-invariant) measure, or Haar measure.

We do not give the proof of this theorem in general. However, all examples we consider are smooth submanifolds in GL_k. Thus, to define the invariant measure we just need to define a volume in the tangent space at identity T_1G and then use right (left) multiplication to define it on the whole group. More precisely, let $\gamma \in \Lambda^{\mathrm{top}} T_1^* G$. Then

$$\gamma_s = m_s^* (\gamma),$$

where $m_s : G \to G$ is the right (left) multiplication on s and m_s^* is the induced map $\Lambda^{\mathrm{top}} T_1^* G \to \Lambda^{\mathrm{top}} T_s^* G$. After this normalize γ to satisfy $\int_G \gamma = 1$.

Consider a vector space over $\mathbb{C}$ equipped with topology such that addition and multiplication by a scalar are continuous. We always assume that a topological vector space satisfies the following conditions

(1) for any $v \in V$ there exist a neighborhood of 0 which does not contain v;
(2) there is a base of convex neighborhoods of zero.

Topological vector spaces satisfying above conditions are called *locally convex.* We do not go into the theory of such spaces. All we need is the fact that there is a non-zero continuous linear functional on a locally convex space.

A representation $\rho : G \to \mathrm{GL}(V)$ is continuous if the map $G \times V \to V$ given by $(s, v) \mapsto \rho_s v$ is continuous.

Regular representation. Let G be a compact group and $L^2(G)$ be the space of all complex valued functions on G such that

$$\int |f(t)|^2 dt$$

exists. Then $L^2(G)$ is a Hilbert space with respect to Hermitian form

$$\langle f, g \rangle = \int_G \bar{f}(t)\, g(t)\, dt.$$

Moreover, a representation R of G in $L^2(G)$ given by

$$R_s f(t) = f(ts)$$

is continuous and the Hermitian form is G-invariant.

A representation $\rho : G \to \mathrm{GL}(V)$ is called *topologically irreducible* if any invariant closed subspace of V is either V or 0.

Lemma 2.2. *Every irreducible representation of G is isomorphic to a subrepresentation in $L^2(G)$.*

Proof. Let $\rho : G \to \mathrm{GL}(V)$ be irreducible. Pick a non-zero linear functional φ on V and define the map $\Phi : V \to L^2(G)$ which sends v to the matrix coefficient $f_{v,\varphi}(s) = \langle \varphi, \rho_s v \rangle$. It is clear that a matrix coefficient is a continuous function on G, therefore $f_{v,\varphi} \in L^2(G)$. Furthermore Φ is a continuous intertwiner and $\mathrm{Ker}\, \Phi = 0$. $\square$

Recall that a Hilbert space is a space over $\mathbb{C}$ equipped with positive definite Hermitian form $\langle , \rangle$ complete in topology defined by the norm

$$\|v\| = \langle v, v \rangle^{1/2}.$$

We need the fact that a Hilbert space has an orthonormal topological basis. A continuous representation $\rho : G \to \mathrm{GL}(V)$ is called *unitary* if V is a Hilbert space and

$$\langle v, v \rangle = \langle \rho_g v, \rho_g v \rangle$$

for any $v \in V$ and $g \in G$. The regular representation of G in $L^2(G)$ is unitary. In fact, Lemma 2.2 implies

Corollary 2.3. *Every topologically irreducible representation of a compact group G is a subrepresentation in $L^2(G)$.*

Lemma 2.4. *Every irreducible unitary representation of a compact group G is finite-dimensional.*

Proof. Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible unitary representation. Choose $v \in V$, $\|v\| = 1$. Define an operator $T : V \to V$ by the formula

$$Tx = \langle v, x \rangle v.$$

One can check easily that T is self-adjoint, i.e.

$$\langle x, Ty \rangle = \langle Tx, y \rangle.$$

Let

$$\bar{T}x = \int_G \rho_g T\left(\rho_g^{-1} x\right) dg.$$

Then $\bar{T} : V \to V$ is an intertwiner and a self-adjoint operator. Furthermore, $\bar{T}$ is compact, i.e. if

$$S = \{x \in V \mid \|x\| = 1\},$$

then $\bar{T}(S)$ is a compact set in V. Every self-adjoint compact operator has an eigenvector. To construct an eigen vector find $x \in S$ such that $|\left(\bar{T}x, x\right)|$ is maximal. Then $\bar{T}x = \lambda x$. Since $\mathrm{Ker}\left(\bar{T} - \lambda\,\mathrm{Id}\right)$ is an invariant subspace in V, $\mathrm{Ker}\left(\bar{T} - \lambda\,\mathrm{Id}\right) = 0$. Hence $\bar{T} = \lambda Id$. Note that for any orthonormal system of vectors $e_1, \dots, e_n \in V$

$$\sum \left\langle e_i, \bar{T} e_i \right\rangle = \sum \langle e_i, T e_i \rangle \leq 1,$$

that implies $\lambda n \leq 1$. Hence $\dim V \leq \frac{1}{\lambda}$. □

Corollary 2.5. *Every irreducible continuous representation of a compact group G is finite-dimensional.*

3. Orthogonality relations and Peter-Weyl Theorem

If $\rho : G \to \mathrm{GL}(V)$ is a unitary representation. Define a matrix coefficient by the formula

$$f_{v,w}(g) = \langle w, \rho_g v \rangle .$$

It is easy to check that

$$f_{v,w}\left(g^{-1}\right) = \bar{f}_{w,v}(g) \tag{3.1}$$

Theorem 3.1. *For an irreducible unitary representation $\rho : G \to \mathrm{GL}(V)$*

$$\langle f_{v,w}, f_{v',w'} \rangle = \int_G \bar{f}_{v,w}(g) f_{v',w'}(g)\, dg = \frac{1}{\dim \rho} \langle v, v' \rangle \langle w', w \rangle .$$

The matrix coefficient of two non-isomorphic representation are orthogonal in $L^2(G)$.

Proof. Define $T \in \mathrm{End}_{\mathbb{C}}(V)$

$$Tx = \langle v, x \rangle v'$$

and

$$\bar{T} = \int_G \rho_g T \rho_g^{-1} dg.$$

As follows from Shur's lemma, $\bar{T} = \lambda Id$. Since

$$\operatorname{tr} \bar{T} = \operatorname{tr} T = \langle v, v' \rangle ,$$

we obtain

$$\bar{T} = \frac{\langle v, v' \rangle}{\dim \rho}.$$

Hence

$$\left\langle w', \bar{T} w \right\rangle = \frac{1}{\dim \rho} \langle v, v' \rangle \langle w', w \rangle .$$

On the other hand,

$$\left\langle w', \bar{T} w \right\rangle = \int_G \left\langle w', \left\langle v, \rho_g^{-1} w \right\rangle \rho_g v' \right\rangle dg = \int_G f_{w,v}\left(g^{-1}\right) f_{v',w'}(g)\, dg =$$

$$= \int_G \bar{f}_{v,w}(g) f_{v',w'}(g)\, dg = \frac{1}{\dim \rho} \langle f_{v,w}, f_{v',w'} \rangle .$$

In $f_{v,w}$ and $f_{v',w'}$ are matrix coefficients of two non-isomorphic representation, the $\bar{T} = 0$, and the calculation is even simpler. ☐

Corollary 3.2. *Let ρ and σ be two irreducible representations, then $\langle \chi_\rho, \chi_\sigma \rangle = 1$ if ρ is isomorphic to σ and $\langle \chi_\rho, \chi_\sigma \rangle = 0$ otherwise.*

Theorem 3.3. *(Peter-Weyl) Matrix coefficient form a dense set in $L^2(G)$ for a compact group G.*

Proof. We will prove the Theorem under assumption that $G \subset \mathrm{GL}(E)$, in other words we assume that G has a faithful finite-dimensional representation. Let $M = \mathrm{End}_{\mathbb{C}}(E)$. The polynomial functions $\mathbb{C}[M]$ on M form a dense set in the space of continuous functions on G (Weierstrass theorem), and continuous functions are dense in $L^2(G)$. On the other hand, $\mathbb{C}[M]$ is spanned by matrix coefficients of all representations in $T(E) = \oplus_{n=0}^{\infty} E^{\otimes n}$. Hence matrix coefficients are dense in $L^2(G)$. □

Corollary 3.4. *The characters of irreducible representations form an orthonormal basis in the subspace of class function in* $L^2(G)$.

Corollary 3.5. *Let G be a compact group and R denote the representation of $G \times G$ in $L^2(G)$ given by the formula*

$$R_{s,t} f(x) = f\left(s^{-1} x t\right).$$

Then

$$L^2(G) \cong \oplus_{\rho \in \widehat{G}} V_\rho \boxtimes V_\rho^*,$$

where $\widehat{G}$ denotes the set of isomorphism classes of irreducible unitary representations of G and the direct sum is in the sense of Hilbert spaces.

Remark 3.6. Note that it follows from the proof of Theorem 3.3, that if E is a faithful representation of a compact group G, then all other irreducible representations appear in $T(E)$ as subrepresentations.

4. Examples

Example 1. Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, $z = e^{i\theta}$. The invariant measure on S^1 is $\frac{d\theta}{2\pi}$ The irreducible representations are one dimensional. They are given by the characters $\chi_n : S^1 \to \mathbb{C}^*$, where $\chi_n(\theta) = e^{in\theta}$. Hence $\widehat{S^1} = \mathbb{Z}$ and

$$L^2\left(S^1\right) = \oplus_{n \in \mathbb{Z}} \mathbb{C} e^{in\theta},$$

this is well-known fact that every periodic function can be extended in Fourier series.

Example 2. Let $G = SU_2$. Then G consists of all matrices

$$\begin{matrix} a & b \\ -\bar{b} & \bar{a}, \end{matrix}$$

satisfying the relations $|a|^2 + |b|^2 = 1$. One also can realize SU_2 as the subgroup of quaternions with norm 1. Thus, topologically SU_2 is isomorphic to the three-dimensional sphere S^3. To find all irreducible representation of SU_2 consider the polynomial ring $\mathbb{C}[x, y]$ with the action of SU_2 given by the formula

$$\rho_g(x) = ax + \text{by},\ \rho_g(y) = -\bar{b}x + \bar{a}y, \text{ if } g = \begin{pmatrix} \frac{a}{-b} \frac{b}{a} \end{pmatrix}.$$

Let ρ_n be the representation of G in the space $\mathbb{C}_n[x,y]$ of homogeneous polynomials of degree n. The monomials x^n, $x^{n-1}y, \dots, y^n$ form a basis of $\mathbb{C}_n[x,y]$. Therefore $\dim \rho_n = n+1$. We claim that all ρ_n are irreducible and that every irreducible representation of SU_2 is isomorphic to ρ_n. Hence $\widehat{G} = \mathbb{Z}_+$. We will show this by checking that the characters χ_n of ρ_n form an orthonormal basis in the Hilbert space of class functions on G.

Note that every unitary matrix is diagonal in some orthonormal basis, therefore every conjugacy class of SU_2 intersects the diagonal subgroup. Moreover, $\left(\begin{smallmatrix} z & 0 \\ 0 & \bar{z} \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \bar{z} & 0 \\ 0 & z \end{smallmatrix}\right)$ are conjugate. Hence the set of conjugacy classes can be identified with S^1 quotient by the equivalence relation $z \sim \bar{z}$. Let $z = e^{i\theta}$, then

$$\chi_n(z) = z^n + z^{n-2} + \dots + z^{-n} = \frac{z^{n+1} - z^{-n-1}}{z - z^{-1}} = \frac{\sin(n+1)\theta}{\sin\theta}. \tag{4.1}$$

Now let us calculate the scalar product in the space of class function. It is clear that the invariant measure dg on G is proportional to the standard volume form on the three-dimensional sphere induced by the volume form on $\mathbb{R}^4$. Let $C(\theta)$ denote the conjugacy class of all matrices with eigenvalues $e^{i\theta}, e^{-i\theta}$. The characteristic polynomial of a matrix from $C(\theta)$ equals $t^2 - 2\cos\theta t + 1$. Thus, we obtain $a + \bar{a} = 2\cos\theta$, or $a = \cos\theta + yi$ for real y. Hence $C(\theta)$ satisfy the equation

$$|a|^2 + |b^2| = \cos^2\theta + y^2 + |b|^2 = 1,$$

or

$$y^2 + |b|^2 = \sin^2\theta.$$

In other words, $C(\theta)$ is a two-dimensional sphere of radius $\sin\theta$. Hence for a class function ϕ on G

$$\int \phi(g)\, dg = \frac{1}{\pi}\int_0^{2\pi} \phi(\theta)\sin^2\theta d\theta.$$

All class function are even functions on S^1, i.e. they satisfy the condition $\phi(-\theta) = \phi(\theta)$. One can see easily from (4.1) that $\chi_n(\theta)$ form an orthonormal basis in the space of even function on the circle with respect to the Hermitian product

$$\langle \varphi, \eta \rangle = \frac{1}{\pi}\int_0^{2\pi} \bar{\varphi}(\theta)\,\eta(\theta)\sin^2\theta d\theta.$$

Example 3. Let $G = SO_3$. Recall that SU_2 can be realized as the set of quaternions with norm 1. Consider the representation γ of SU_2 in $\mathbb{H}$ defined by the formula $\gamma_g(\alpha) = g\alpha g^{-1}$. One can see that the 3-dimensional space $\mathbb{H}_{\text{im}}$ of pure imaginary quaternions is invariant and $(\alpha, \beta) = \operatorname{Re}\left(\alpha\bar{\beta}\right)$ is invariant positive definite scalar product on $\mathbb{H}_{\text{im}}$. Therefore ρ defines a homomorphism $\gamma\colon SU_2 \to SO_3$. Check that $\operatorname{Ker}\gamma = \{1, -1\}$ and that γ is surjetive. Hence $SO_3 \cong SU_2/\{1,-1\}$. Thus, every representation of SO_3 can be lifted to the representations of SU_2, and a representation of SU_2 factors to the representation of SO_3 iff it is trivial on -1. One can check easily that $\rho_n(-1) = 1$ iff n is even. Thus, an irreducible representations of SO_3 is isomorphic to ρ_{2m} and $\dim \rho_{2m} = 2m+1$. Below we give an independent realization of irreducible representation of SO_3.

Harmonic analysis on a sphere. Consider the sphere S^2 in $\mathbb{R}^3$ defined by the equation $x^2+y^2+z^2=1$. It is clear that SO_3 acts in the space of complex-valued functions on S^2. Introduce differential operators in $\mathbb{R}^3$:

$$e = \frac{-1}{2}\left(x^2+y^2+z^2\right),\ h = x\partial_x + y\partial_y + z\partial_z + \frac{3}{2},\ f = \frac{1}{2}\left(\partial_x^2+\partial_y^2+\partial_z^2\right),$$

note that e, f, and h commute with the action of SO_3 and satisfy the relations

$$[e,f]=h,\ [h,e]=2e,\ [h,f]=-2f.$$

Let P_n be the space of homogeneous polynomial of degree n and $H_n = \operatorname{Ker} f \cap P_n$. The polynomials of H_n are harmonic polynomials since they are annihilated by Laplace operator. For any $\varphi \in P_n$

$$h(\varphi) = \left(n+\frac{3}{2}\right)\varphi.$$

If $\varphi \in H_n$, then

$$fe(\varphi) = ef(\varphi) - h(\varphi) = -\left(n+\frac{3}{2}\right)\varphi,$$

and by induction

$$fe^k(\varphi) = efe^{k-1}(\varphi) - he^{k-1}(\varphi) = -\left(nk + k(k-1) + \frac{3k}{2}\right)e^{k-1}\varphi.$$

In particular, this implies that

$$fe^k(H_n) = e^{k-1}(H_n). \tag{4.2}$$

We prove that

$$P_n = H_n \oplus e(H_{n-2}) \oplus e^2(H_{n-4}) + \dots \tag{4.3}$$

by induction on n. Indeed, by induction assumption

$$P_{n-2} = H_{n-2} \oplus e(H_{n-4}) + \dots,$$

then (4.2) implies $fe(P_{n-2}) = P_{n-2}$. Hence $H_n \cap eP_{n-2} = 0$. On the other hand, $f\colon P_n \to P_{n-2}$ is surjective, and therefore $\dim H_n + \dim P_{n-2} = \dim P_n$. Therefore

$$P_n = H_n \oplus P_{n-2}, \tag{4.4}$$

which implies (4.3). Note that after restriction on S^2, the operator e acts as the multiplication on $\frac{-1}{2}$.

Hence (4.3) implies that

$$\mathbb{C}\left[S^2\right] = \oplus_{n\geq 0} H_n.$$

To calculate the dimension of H_n use (4.4)

$$\dim H_n = \dim P_n - \dim P_{n-2} = \frac{(n+1)(n+2)}{2} - \frac{n(n-1)}{2} = 2n+1.$$

Finally, we claim that the representation of SO_3 in H_n is irreducible and isomorphic to ρ_{2n}. Check that $\varphi = (x+iy)^n \in H_n$ and the rotation on the angle θ about $z-$ axis maps φ to $e^{in\theta}\varphi$. Since this rotation is the image of

$$\begin{matrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2}, \end{matrix}$$

under the homomorphism $\gamma : SU_2 \to SO_3$, the statement follows from (4.1).

Recall now the following theorem (Lecture Notes 1).

A convex centrally symmetric solid in $\mathbb{R}^3$ is uniquely determined by the areas of the plane cross-sections through the origin.

A convex solid B can be defined by an even continuous function on S^2. Indeed, for each unit vector v let

$$\varphi(v) = \sup\left\{t^2/2 \mid tv \in B\right\}.$$

Define a linear operator T in the space of all even continuous functions on S^2 by the formula

$$T\varphi(v) = \int_0^{2\pi} \varphi(w)\, d\theta,$$

where w runs the set of unit vectors orthogonal to v, and θ is the angular parameter on the circle $S^2 \cap v^{\perp}$. Check that $T\varphi(v)$ is the area of the cross section by the plane $v^{\perp}$. We have to prove that T is invertible.

Obviously T commutes with the SO_3-action. Therefore T can be diagonalized. Moreover, T acts on H_{2n} as the scalar operator $\lambda_n Id$. We have to check that $\lambda_n \neq 0$ for all n. Let $\varphi = (x+iy)^{2n} \in H_{2n}$. Then $\varphi(1,0,0) = 1$ and

$$T\varphi(1,0,0) = \int_0^{2\pi} (iy)^{2n}\, d\theta = (-1)^n \int_0^{2\pi} \sin^{2n}\theta d\theta,$$

here we take the integral over the circle $y^2 + z^2 = 1$, and assume $y = \sin\theta$, $z = \cos\theta$. Since $T\varphi = \lambda_n\varphi$, we obtain

$$\lambda_n = (-1)^n \int_0^{2\pi} \sin^{2n}\theta d\theta \neq 0.$$

PROBLEM SET # 7
MATH 252

Due October 21.

1. Show that SO_4 is isomorphic to the quotient of $SU_2 \times SU_2$ by the subgroup generated by $(-1,-1)$. Hint : consider the representation of $SU_2 \times SU_2$ in the space of quaternions $\mathbb{H}$ by left and right multiplication.

2. Show the following identity for representations of SU_2

$$\rho_m \otimes \rho_n = \rho_{m+n} \oplus \rho_{m+n-2} \oplus \cdots \oplus \rho_{m-n},$$

assuming $m \geq n$.

Date: October 13, 2005.

REPRESENTATION THEORY. WEEK 8

VERA SERGANOVA

1. Representations of $\mathrm{SL}_2(\mathbb{R})$

In this section
$$G = \mathrm{SL}_2(\mathbb{R}) = \{g \in \mathrm{GL}_2(\mathbb{R}) \mid \det g = 1\}.$$
Let K be the subgroup of matrices
$$g_\theta = \begin{pmatrix} \cos & \theta & \sin & \theta \\ -\sin & \theta & \cos & \theta \end{pmatrix}.$$
The group K is a maximal compact subgroup of G, clearly K is isomorphic to S^1. If $\rho\colon G \to \mathrm{GL}(V)$ is a unitary representation of G in a Hilbert space then then $\mathrm{Res}_K \rho$ splits into the sum of 1-dimensional representations of V. In particular, one can find $v \in V$ such that $\rho_{g_\theta}(v) = e^{in\theta}v$. Define the matrix coefficient function $f : G \to \mathbb{C}$ given by
$$f(g) = \langle v, \rho_g v\rangle.$$
Then f satisfies the condition
$$f(gg_\theta) = e^{in\theta} f(g).$$
Thus, one can consider f as a section of a linear bundle on the space G/K (if $n = 0$, then f is a function). Thus, it is clear that the space G/K is an important geometric object, where the representations of G are "realized".

Consider the Lobachevsky plane
$$H = \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\}$$
with metric defined by the formula $\frac{dx^2+dy^2}{y^2}$ and the volume form $\frac{dxdy}{y^2}$. Then G coincides with the group of rigid motions of H preserving orientation. The action of G on H is given by the formula
$$z \mapsto \frac{az+b}{cz+d}.$$
One can check easily that G acts transitively on H, preserves the metric and volume. Moreover, the stabilizer of $i \in H$ coincides with K. Thus, we identify H with G/K.

The first series of representations we describe is called the representations of *discrete series*. Those are the representations with matrix coefficients in $L^2(G)$. Let

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$\mathcal{H}_n^+$ be the space of holomorphic densities on H, the expressions $\varphi(z)(dz)^{n/2}$, where $\varphi(z)$ is a holomorphic function on H satisfying the condition that

$$\int |\varphi|^2 y^{n-2} dz d\bar{z}$$

is finite. Define the representation of G in $\mathcal{H}_n^+$ by

$$\rho_g\left(\varphi(z)(dz)^{n/2}\right) = \varphi\left(\frac{az+b}{cz+d}\right)\frac{1}{(cz+d)^n}(dz)^{n/2},$$

and Hermitian product on $\mathcal{H}_n$ the formula

$$\left\langle \varphi(dz)^{n/2}, \psi(dz)^{n/2}\right\rangle = \int \bar{\varphi}\psi y^{n-2} dz d\bar{z}, \tag{1.1}$$

for $n > 1$. For $n = 1$ the product is defined by

$$\left\langle \varphi(dz)^{n/2}, \psi(dz)^{n/2}\right\rangle = \int_{-\infty}^{\infty} \bar{\varphi}\psi dx, \tag{1.2}$$

in this case $\mathcal{H}_1^+$ consists of all densities which converge to L^2-functions on the boundary (real line). Check that this Hermitian product is invariant.

Let us show that $\mathcal{H}_n$ is irreducible. It is convenient to consider Poincaré model of Lobachevsky plane using the conformal map

$$w = \frac{z-i}{z+i},$$

that maps H to a unit disk $|w| < 1$. Then the group G acts on the unit disk by linear-fractional maps $w \to \frac{aw+b}{\bar{b}w+\bar{a}}$ for all complex a, b

satisfying $|a|^2 - |b|^2 = 1$, and K is defined by the condition $b = 0$. If $a = e^{i\theta}$, then $\rho_{g_\theta}(w) = e^{2i\theta}w$. The invariant volume form is $\frac{dwd\bar{w}}{1-\bar{w}w}$.

It is clear that $w^k(dw)^{n/2}$ for all $k \geq 0$ form an orthogonal basis in $\mathcal{H}_n^+$, each vector $w^k(dw)^{n/2}$ is an eigen vector with respect to K, namely

$$\rho_{g_\theta}\left(w^k(dw)^{n/2}\right) = e^{(2k+n)i\theta}w^k(dw)^{n/2}.$$

It is easy to check now that $\mathcal{H}_n^+$ is irreducible. Indeed, every invariant closed subspace V has a topological basis consisting of eigenvectors of K, in other words $w^k(dw)^{n/2}$ for some positive k must form a topological basis of V. Without loss of generality assume that V contains $(dw)^{n/2}$, then by applying ρ_g one can get that $\frac{1}{(bw+a)^n}(dw)^{n/2}$, and in Taylor series for $\frac{1}{(bw+a)^n}$ all elements of the basis appear with non-zero coefficients. That implies $w^k(dw)^{n/2} \in V$ for all $k \geq 0$, hence $V = \mathcal{H}_n^+$.

One can construct another series of representations $\mathcal{H}_n^-$ by considering holomorphic densities in the lower half-plane $\operatorname{Im} z < 0$.

Principal series. These representations are parameterized by a continuous parameter $s \in \mathbb{R}i\,(s \neq 0)$. Consider now the action of G on a real line by linear fractional transformations $x \mapsto \frac{ax+b}{cx+d}$. Let $\mathcal{P}_s^+$ denotes the space of densities $\varphi(x)(dx)^{\frac{1+s}{2}}$ with G-action given by

$$\rho_g\left(\varphi(x)(dx)^{\frac{1+s}{2}}\right) = \varphi\left(\frac{ax+b}{cx+d}\right)|cx+d|^{-s-1}(dx)^{\frac{1+s}{2}}.$$

The Hermitian product given by

$$\langle\varphi,\psi\rangle = \int_{-\infty}^{\infty}\bar{\varphi}\psi dx \tag{1.3}$$

is invariant. The property of invariance justify the choice of weight for the density as $(dx)^{\frac{1+s}{2}}(dx)^{\frac{1+\bar{s}}{2}} = dx$, thus the integration is invariant. To check that the representation is irreducible one can move the real line to the unit circle as in the example of discrete series and then use $e^{ik\theta}(d\theta)^{\frac{1+s}{2}}$ as an orthonormal basis in $\mathcal{P}_s^+$. Note that the eigen values of ρ_{g_θ} in this case are $e^{2ki\theta}$ for all integer k.

The second principal series $\mathcal{P}_s^-$ can be obtained if instead of densities we consider the pseudo densities which are transformed by the law

$$\rho_g\left(\varphi(x)(dx)^{\frac{1+s}{2}}\right) = \varphi\left(\frac{ax+b}{cx+d}\right)|cx+d|^{-s-1}\operatorname{sgn}(cx+d)\,dx^{\frac{1+s}{2}}.$$

Complementary series. Those are representations which do not appear in the regular representation $L^2(G)$. They can be realized as the representations in $\mathcal{C}_s$ of all densities $\varphi(x)(dx)^{\frac{1+s}{2}}$ for real $0<s<1$. An invariant Hermitian product is

$$\langle\varphi,\psi\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{\varphi}(x)\psi(y)|x-y|^{s-1}dxdy. \tag{1.4}$$

2. Semisimple modules and density theorem

We assume that R is a unital ring. Recall that an R-module is *semi-simple* if for any submodule $N\subset M$ there exists a submodule N' such that $M=N\oplus N'$ and R-module M is *simple* if any submodule of M is either M or 0.

Lemma 2.1. *Every submodule and every quotient of a semisimple module is semisimple.*

Proof. If Let N be a submodule of a semisimple module M, and let P be a submodule of N. Since $M=P\oplus P'$, then there exists an R-invariant projector $p: M\to P$. The restriction of p to N defines the projector $N\to P$. □

Lemma 2.2. *Any semisimple R-module contains a simple submodule.*

Proof. Let M be semisimple, $m\in M$. Let N be a maximal submodule in Rm which does not contain m (exists by Zorn's lemma, take all submodules which do not contain m). Then Rm is semisimple and $Rm=N\oplus N'$. We claim that N' is simple. Indeed, if N' is not simple, then it contains a proper submodule P. But $m\notin P\oplus N$, since $P\oplus N\neq Rm$. That contradicts maximality of N. □

Lemma 2.3. *The following conditions on a module M are equivalent*

1. *M is semisimple;*
2. *$M = \sum_{i \in I} M_i$ for some simple submodules M_i;*
3. *$M = \oplus_{j \in J} M_j$ for some simple submodules M_j.*

Proof. (1) $\Rightarrow$ (2) Let $\{M_i\}_{i \in I}$ be the collection of all simple submodules. Let $N = \sum_{i \in I} M_i$, assume that $N \neq M$, then $M = N \oplus N'$ and N' contains a simple submodule. Contradiction.

To prove (2) $\Rightarrow$ (3) let $J \subset I$ be minimal such that $M = \sum_{j \in J} M_j$ (check that it exists by Zorn's lemma). By minimality of J for any $k \in J$, M_k does not belong to $\sum_{j \in J - k} M_j$. Therefore $M = \oplus_{j \in J} M_j$.

Finally, let us prove (3) $\Rightarrow$ (1). Let $N \subset M$ be a submodule and $S \subset J$ be a maximal subset such that $N \cap \oplus_{j \in S} M_j = 0$. Let $M' = N \oplus (\oplus_{j \in S} M_j)$. We claim that $M' = M$. Indeed, assume that the statement is false. Then there exists k such that $M_k \cap M' = 0$. But then $N \cap \oplus_{j \in S+k} M_j = 0$. Contradiction. □

Lemma 2.4. *Let M be a semisimple module. Then M is simple iff $\mathrm{End}_R(M)$ is a division ring.*

Proof. In one direction this is Shur's lemma. In the opposite direction if $M = M_1 \oplus M_2$, then the projectors p_1, p_2 satisfy $p_1 p_2 = 0$ and therefore p_1, p_2 are not invertible. □

Lemma 2.5. *Let $\mathrm{End}_R(M) = K$, $\mathrm{End}_K(M) = S$. Then $\widehat{K} = \mathrm{End}_R(M^{\oplus n}) \cong \mathrm{Mat}_n(K)$ and $\mathrm{End}_{\widehat{K}}(M^{\oplus n}) \cong S$, the last isomorphism is given by the diagonal action*

$$s(v_1, \dots, v_n) = (sv_1, \dots, sv_n).$$

Proof. See similar statement in lecture notes 3. □

Theorem 2.6. *(Jacobson-Chevalley density theorem). Let M be a semisimple R-module, $K = \mathrm{End}_R(M)$, $S = \mathrm{End}_K(M)$. Then for any $v_1, \dots, v_n \in M$ and $X \in S$ there exists $r \in R$ such that $rv_i = Xv_i$ for all $i = 1, \dots, n$.*

Proof. First, let us prove it for $n = 1$. It suffices to show that Rv is S-invariant. Indeed, $M = Rv \oplus N$, and p be the projector on N with kernel Rv. Then $p \in K$, hence $\mathrm{Ker}\, p$ is S-invariant.

For arbitrary n, note that $M^{\oplus n}$ is semisimple and use Lemma 2.5. Then for any $X \in S$, $v = (v_1, \dots, v_n)$ there exists $r \in R$ such that

$$r(v_1, \dots, v_n) = X(v_1, \dots, v_n).$$

□

Corollary 2.7. *Let M be a semisimple R-module, $K = \mathrm{End}_R(M)$, and M is finitely generated over K. Then the natural map $R \to \mathrm{End}_K(M)$ is surjective.*

Corollary 2.8. *Let R be an algebra over an algebraically closed field k, and $\rho \colon R \to \mathrm{End}_k(V)$ be an irreducible finite-dimensional representation. Then ρ is surjective.*

Corollary 2.9. *Let R, ρ and V be as in previous statement but k is not algebraically closed. Then $\rho(R) \cong \mathrm{End}_D(V)$ for some division ring D containing k.*

3. Semisimple rings

A ring R is *semi-simple* if every R-module is semisimple. For example, a group algebra $k(G)$ for a finite group G such that $\mathrm{char}\, k$ does not divide $|G|$ is semisimple.

Lemma 3.1. *Let R be semisimple, then R is a finite direct sum of its minimal left ideals.*

Proof. R is semisimple over itself. Hence $R = \oplus_{i \in I} L_i$, where L_i are minimal left ideals. But $1 = l_1 + \cdots + l_n$, hence I is finite. □

Theorem 3.2. *(Wedderburn-Artin) Every semisimple ring is isomorphic to $\mathrm{Mat}_{n_1}(D_1) \times \cdots \times \mathrm{Mat}_{n_k}(D_k)$ for some division rings $D_1, \dots, D_k$.*

Proof. Consider the submodules $J_1, \dots, J_k$, each J_i is the sum of all isomorphic minimal left ideals. Then J_i is a two-sided ideal and

$$R = J_1 \oplus \cdots \oplus J_k,$$

where $J_i = l_i^{\oplus n_i}$. Let $D_i^{\mathrm{op}} = \mathrm{End}_R(l_i)$, then

$$R^{\mathrm{op}} \cong \mathrm{End}_R(R) = \mathrm{End}_R(J_1) \times \cdots \times \mathrm{End}_R(J_k) = \mathrm{Mat}_{n_1}(D_1^{\mathrm{op}}) \times \cdots \times \mathrm{Mat}_{n_k}(D_k^{\mathrm{op}}).$$

□

PROBLEM SET # 8
MATH 252

Due October 28.

1. Classify all irreducible (continuous) representations of O_2 (the group of orthogonal 2×2-matrices).

2. Check that the Hermitian products (1.2) and (1.4) defined in lecture notes 8 are invariant.

3. Show that the operator $T\colon \mathcal{P}_s^+ \to \mathcal{P}_{-s}^+$ defined by the formula

$$T\left(\phi\left(x\right)dx^{\frac{1+s}{2}}\right) = \left(\int_{-\infty}^{\infty} \phi\left(x\right)\left|x-y\right|^{s-1}dx\right)dy^{\frac{1-s}{2}}$$

is intertwining. Therefore $\mathcal{P}_s^+$ is isomorphic to $\mathcal{P}_{-s}^+$.

Date: October 20, 2005.

REPRESENTATION THEORY
WEEK 9

1. Jordan-Hölder theorem and indecomposable modules

Let M be a module satisfying ascending and descending chain conditions (ACC and DCC). In other words every increasing sequence submodules $M_1 \subset M_2 \subset \dots$ and any decreasing sequence $M_1 \supset M_2 \supset \dots$ are finite. Then it is easy to see that there exists a finite sequence

$$M = M_0 \supset M_1 \supset \cdots \supset M_k = 0$$

such that M_i/M_{i+1} is a simple module. Such a sequence is called a Jordan-Hölder series. We say that two Jordan Hölder series

$$M = M_0 \supset M_1 \supset \cdots \supset M_k = 0,\ M = N_0 \supset N_1 \supset \cdots \supset N_l = 0$$

are equivalent if $k = l$ and for some permutation s $M_i/M_{i+1} \cong N_{s(i)}/N_{s(i)+1}$.

Theorem 1.1. *Any two Jordan-Hölder series are equivalent.*

Proof. We will prove that if the statement is true for any submodule of M then it is true for M. (If M is simple, the statement is trivial.) If $M_1 = N_1$, then the statement is obvious. Otherwise, $M_1 + N_1 = M$, hence $M/M_1 \cong N_1/(M_1 \cap N_1)$ and $M/N_1 \cong M_1/(M_1 \cap N_1)$. Consider the series

$$M = M_0 \supset M_1 \supset M_1 \cap N_1 \supset K_1 \supset \cdots \supset K_s = 0,\ M = N_0 \supset N_1 \supset N_1 \cap M_1 \supset K_1 \supset \cdots \supset K_s = 0.$$

They are obviously equivalent, and by induction assumption the first series is equivalent to $M = M_0 \supset M_1 \supset \cdots \supset M_k = 0$, and the second one is equivalent to $M = N_0 \supset N_1 \supset \cdots \supset N_l = \{0\}$. Hence they are equivalent. □

Thus, we can define a length $l(M)$ of a module M satisfying ACC and DCC, and if M is a proper submodule of N, then $l(M) < l(N)$.

A module M is *indecomposable* if $M = M_1 \oplus M_2$ implies $M_1 = 0$ or $M_2 = 0$.

Lemma 1.2. *Let M and N be indecomposable, $\alpha \in \mathrm{Hom}_R(M, N)$, $\beta \in \mathrm{Hom}_R(N, M)$ be such that $\beta \circ \alpha$ is an isomorphism. Then α and β are isomorphisms.*

Proof. We claim that $N = \mathrm{Im}\,\alpha \oplus \mathrm{Ker}\,\beta$. Indeed, $\mathrm{Im}\,\alpha \cap \mathrm{Ker}\,\beta = 0$ and for any $x \in N$ one can write $x = y + z$, where $y = \alpha \circ (\beta \circ \alpha)^{-1} \circ \beta(x)$, $z = x - y$. Then since N is indecomposable, $\mathrm{Im}\,\alpha = N$, $\mathrm{Ker}\,\beta = 0$ and $N \cong M$. □

Date: November 7, 2005.

Lemma 1.3. *Let M be indecomposable module of finite length and $\varphi \in \operatorname{End}_R(M)$, then either φ is an isomorphism or φ is nilpotent.*

Proof. There is $n > 0$ such that $\operatorname{Ker} \varphi^n = \operatorname{Ker} \varphi^{n+1}$, $\operatorname{Im} \varphi^n = \operatorname{Im} \varphi^{n+1}$. In this case $\operatorname{Ker} \varphi^n \cap \operatorname{Im} \varphi^n = 0$ and hence $M \cong \operatorname{Ker} \varphi^n \oplus \operatorname{Im} \varphi^n$. Either $\operatorname{Ker} \varphi^n = 0$, $\operatorname{Im} \varphi^n = M$ or $\operatorname{Ker} \varphi^n = M$. Hence the lemma. □

Lemma 1.4. *Let M be as in Lemma 1.3 and $\varphi, \varphi_1, \varphi_2 \in \operatorname{End}_R(M)$, $\varphi = \varphi_1 + \varphi_2$. If φ is an isomorphism then at least one of φ_1, φ_2 is also an isomorphism.*

Proof. Without loss of generality we may assume that $\varphi = id$. But in this case φ_1 and φ_2 commute. If both φ_1 and φ_2 are nilpotent, then $\varphi_1 + \varphi_2$ is nilpotent, but this is impossible as $\varphi_1 + \varphi_2 = \mathrm{id}$. □

Corollary 1.5. *Let M be as in Lemma 1.3. Let $\varphi = \varphi_1 + \cdots + \varphi_k \in \operatorname{End}_R(M)$. If φ is an isomorphism then φ_i is an isomorphism at least for one i.*

It is obvious that if M satisfies ACC and DCC then M has a decomposition

$$M = M_1 \oplus \cdots \oplus M_k,$$

where all M_i are indecomposable.

Theorem 1.6. *(Krull-Schmidt) Let M be a module of finite length and*

$$M = M_1 \oplus \cdots \oplus M_k = N_1 \oplus \cdots \oplus N_l$$

for some indecomposable M_i and N_j. Then $k = l$ and there exists a permutation s such that $M_i \cong N_{s(j)}$.

Proof. Let $p_i : M_1 \to N_i$ be the restriction to M_1 of the natural projection $M \to N_i$, and $q_j : N_j \to M_1$ be the restriction to N_j of the natural projection $M \to M_1$. Then obviously $q_1 p_1 + \cdots + q_l p_l = \mathrm{id}$, and by Corollary 1.5 there exists i such that $q_i p_i$ is an isomorphism. Lemma 1.2 implies that $M_1 \cong N_i$. Now one can easily finish the proof by induction on k. □

2. Some facts from homological algebra

The complex is the graded abelian group $C_\cdot = \oplus_{i \geq 0} C_i$. We will assume later that all C_i are R-modules for some ring R. A differential is an R-morphism of degree -1 such that $d^2 = 0$. Usually we realize $C_\cdot$ by the picture

$$\xrightarrow{d} \cdots \to C_1 \xrightarrow{d} C_0 \to 0.$$

We also consider d of degree 1, in this case the superindex $C^\cdot$ and

$$0 \to C^0 \xrightarrow{d} C^1 \xrightarrow{d} \ldots$$

All the proofs are similar for these two cases.

Homology group is $H_i(C) = (\operatorname{Ker} d \cap C_i) / dC_{i+1}$.

Given two complexes $(C_., d)$ and $(C'_., d')$. A morphism $f : C_. \to C'_.$ preserving grading and satisfying $f \circ d = d' \circ f$ is called a *morphism of complexes*. A morphism of complexes induces the morphism $f_* \colon H_.(C) \to H_.(C')$.

Theorem 2.1. *(Long exact sequence). Let*

$$0 \to C_. \xrightarrow{g} C'_. \xrightarrow{f} C''_. \to 0$$

be a short exact sequence, then the long exacts sequence

$$\xrightarrow{\delta} H_i(C) \xrightarrow{g_*} H_i(C') \xrightarrow{f_*} H_i(C'') \xrightarrow{\delta} H_{i-1}(C) \xrightarrow{g_*} \dots$$

where $\delta = g^{-1} \circ d' \circ f^{-1}$, is exact.

Let $f, g : C_. \to C'_.$ be two morphisms of complexes. We say that f and g are *homotopically equivalent* if there exists $h : C_. \to C'_.(+1)$ (the morphism of degree 1) such that $f - g = h \circ d + d' \circ h$.

Lemma 2.2. *If f and g are homotopically equivalent then $f_* = g_*$.*

Proof. Let $\phi = f - g$, $x \in C_i$ and $dx = 0$. Then

$$\phi(x) = h(dx) + d'(hx) = d'(hx) \in \operatorname{Im} d'.$$

Hence $f_* - g_* = 0$. □

We say that complexes $C_.$ and $C'_.$ are homotopically equivalent if there exist $f : C_. \to C'_.$ and $g : C'_. \to C_.$ such that $f \circ g$ is homotopically equivalent to $\mathrm{id}_{C'}$ and $g \circ f$ is homotopically equivalent to id_C. Lemma 2.2 implies that homotopically equivalent complexes have isomorphic homology. The following Lemma is straightforward.

Lemma 2.3. *If $C_.$ and $C'_.$ are homotopically equivalent then the complexes $\operatorname{Hom}_R(C_., B)$ and $\operatorname{Hom}_R(C'_.)$ are also homotopically equivalent.*

Note that the differential in $\operatorname{Hom}_R(C_., B)$ has degree 1.

3. Projective modules

An R-module P is *projective* if for any surjective morphism $\phi : M \to N$ and any $\psi : P \to N$ there exists $f : P \to M$ such that $\psi = \phi \circ f$.

Example. A free module is projective. Indeed, let $\{e_i\}_{i \in I}$be the set of free generators of a free module F, i.e. $F = \oplus_{i \in I} Re_i$. Define $f : F \to M$ by $f(e_i) = \phi^{-1}(\psi(e_i))$.

Lemma 3.1. *The following conditions on a module P are equivalent*

1. *P is projective;*
2. *There exists a free module F such that $F \cong P \oplus P'$;*
3. *Any exact sequence $0 \to N \to M \to P \to 0$ splits.*

Proof. (1) $\Rightarrow$ (3) Consider the exact sequence

$$0 \to N \xrightarrow{\varphi} M \xrightarrow{\psi} P \to 0,$$

then since ψ is surjective, there exists $f : P \to M$ such that $\psi \circ f = \mathrm{id}_P$.

(3) $\Rightarrow$ (2) Every module is a quotient of a free module. Therefore we just have to apply (3) to the exact sequence

$$0 \to N \to F \to P \to 0$$

for a free module F.

(2) $\Rightarrow$ (1) Let $\phi : M \to N$ be surjective and $\psi : P \to N$. Choose a free module F so that $F = P \oplus P'$. Then extend ψ to $F \to N$ in the obvious way and let $f : F \to M$ be such that $\phi \circ f = \psi$. Then the last identity is true for the restriction of f to P. □

A *projective resolution* of M is a complex $P_.$ of projective modules such that $H_i(P_.) = 0$ for $i > 0$ and $H_0(P_.) \cong M$. A projective resolution always exists since one can easily construct a resolution by free modules. Below we prove the "uniqueness" statement.

Lemma 3.2. *Let $P_.$ and $P'_.$ be two projective resolutions of the same module M. Then there exists a morphism $f : P_. \to P'_.$ of complexes such that $f_* : H_0(P_.) \to H_0(P'_.)$ induces the identity id_M. Any two such morphisms f and g are homotopically equivalent.*

Proof. Construct f inductively. Let $p : P_0 \to M$ and $p' : P'_0 \to M$ be the natural projections, define $f : P_0 \to P'_0$ so that $p' \circ f = p$. Then

$$f(\operatorname{Ker} p) \subset \operatorname{Ker} p', \ \operatorname{Ker} p = d(P_1), \ \operatorname{Ker} p' = d'(P'_1),$$

hence $f \circ d(P_1) \subset d'(P'_1)$, and one can construct $f : P_1 \to P'_1$ such that $f \circ d = d' \circ f$. Proceed in the same manner to construct $f : P_i \to P_i$.

To check the second statement, let $\varphi = f - g$. Then $p' \circ \varphi = 0$. Hence

$$\varphi(P_0) \subset \operatorname{Ker} p' = d'(P'_1).$$

Therefore one can find $h : P_0 \to P'_1$ such that $d' \circ h = \varphi$. Furthermore,

$$d' \circ h \circ d = \varphi \circ d = d' \circ \varphi,$$

hence

$$(\varphi - h \circ d)(P_1) \subset P'_1 \cap \operatorname{Ker} d' = d'(P'_2).$$

Thus one can construct $h : P_1 \to P'_2$ such that $d' \circ h = \varphi - h \circ d$. Then proceed inductively to define $h : P_i \to P'_{i+1}$. □

Corollary 3.3. *Every two projective resolutions of M are homotopically equivalent.*

Let M and N be two modules and $P_\cdot$ be a projective resolution of M. Consider the complex

$$0 \to \mathrm{Hom}_R(P_0, N) \to \mathrm{Hom}_R(P_1, N) \to \dots,$$

where the differential is defined naturally. The cohomology of this complex is denoted by $\mathrm{Ext}^{\cdot}_R(M,N)$. Lemma 2.3 implies that $\mathrm{Ext}^{\cdot}_R(M,N)$ does not depend on a choice of projective resolution for M. Check that $\mathrm{Ext}^0_R(M,N) = \mathrm{Hom}_R(M,N)$.

Example 1. Let $R = \mathbb{C}[x]$ be the polynomial ring. Any simple R-module is one-dimensional and isomorphic to $\mathbb{C}[x]/(x-\lambda)$. Denote such module by $\mathbb{C}_\lambda$. A projective resolution of $\mathbb{C}_\lambda$ is

$$0 \to \mathbb{C}[x] \xrightarrow{d} \mathbb{C}[x] \to \ 0,$$

where $d(1) = x - \lambda$. Let us calculate $\mathrm{Ext}^{\cdot}(\mathbb{C}_\lambda, \mathbb{C}_\mu)$. Note that $\mathrm{Hom}_{\mathbb{C}[x]}(\mathbb{C}_\mu) = \mathbb{C}$, hence we have the complex

$$0 \to \mathbb{C} \xrightarrow{d^*} \mathbb{C} \to 0$$

where $d^* = \lambda - \mu$. Hence $\mathrm{Ext}^{\cdot}(\mathbb{C}_\lambda, \mathbb{C}_\mu) = 0$ if $\lambda \neq \mu$ and $\mathrm{Ext}^0(\mathbb{C}_\lambda, \mathbb{C}_\lambda) = \mathrm{Ext}^1(\mathbb{C}_\lambda, \mathbb{C}_\lambda) = \mathbb{C}$.

Example 2. Let $R = \mathbb{C}[x]/(x^2)$. Then R has one up to isomorphism simple module, denote it by $\mathbb{C}_0$. A projective resolution for $\mathbb{C}_0$ is

$$\dots \xrightarrow{d} R \xrightarrow{d} R \to \ 0,$$

where $d(1) = x$ and $\mathrm{Ext}^i(\mathbb{C}_0, \mathbb{C}_0) = \mathbb{C}$ for all $i \geq 0$.

4. Representations of artinian rings

An *artinian* ring is a unital ring satisfying the descending chain condition for left ideals. We will see that an artinian ring is a finite length module over itself. Therefore R is automatically noetherian. A typical example of an artinian ring is a finite-dimensional algebra over a field.

Theorem 4.1. *Let R be an artinian ring, $I \subset R$ be a left ideal. If I is not nilpotent, then I contains an idempotent.*

Proof. Let J be a minimal left ideal, such that $J \subset I$ and J is not nilpotent. Then $J^2 = J$. Let L be a minimal left ideal such that $L \subset J$ and $JL \neq 0$. Then there is $x \in L$ such that $Jx \neq 0$. But then $Jx = L$ by minimality of L. Thus, for some $r \in R$, $rx = x$, hence $r^2x = rx$ and $(r^2 - r)x = 0$. Let $N = \{y \in J \mid yx = 0\}$. Then N is a proper left ideal in J and therefore N is nilpotent. Thus, we obtain

$$r^2 \equiv r \mod N.$$

Let $n = r^2 - r$, then

$$(r + n - 2rn)^2 \equiv r^2 + 2rn - 4r^2n \mod N^2,$$
$$r^2 + 2rn - 4r^2n \equiv r + n - 2rn \mod N^2.$$

Hence $r_1 = r + n - 2rn$ is an idempotent modulo N^2. Repeating this process several times we obtain an idempotent. □

Corollary 4.2. *If an artinian ring does not have nilpotent ideals, then it is semisimple.*

Proof. The sum S of all minimal left ideals is semisimple. By DCC S is a finite direct sum of minimal left ideals. Then S contains an idempotent e, which is the sum of idempotents in each direct summand. Then $R = S \oplus R(1-e)$, however that implies $R = S$. □

Important notion for a ring is the *radical*. For an R-module M let

$$\operatorname{Ann} M = \{x \in R \mid xM = 0\}.$$

Then the radical $\operatorname{rad} R$ is the intersection of $\operatorname{Ann} M$ for all simple R-modules M.

Theorem 4.3. *If R is artinian then $\operatorname{rad} R$ is a maximal nilpotent ideal.*

Proof. First, let us show that $\operatorname{rad} R$ is nilpotent. Assume the contrary. Then $\operatorname{rad} R$ contains an idempotent e. But then e does not act trivially on a simple quotient of Re. Contradiction.

Now let us show that any nilpotent ideal N lies in $\operatorname{rad} R$. Let M be a simple module, then $NM \neq M$ as N is nilpotent. But NM is a submodule of M. Therefore $NM = 0$. Hence $N \subset \operatorname{Ann} M$ for any simple M. □

Corollary 4.4. *An artinian ring R is semisimple iff $\operatorname{rad} R = 0$.*

Corollary 4.5. *If R is artinian, then $R/\operatorname{rad} R$ is semisimple.*

Corollary 4.6. *If R is artinian and M is an R-module, then for the filtration*

$$M \supset (\operatorname{rad} R)\, M \supset (\operatorname{rad} R)^2\, M \supset \cdots \supset (\operatorname{rad} R)^k\, M = 0$$

all quotients are semisimple. In particular, M always has a simple quotient.

Theorem 4.7. *If R is artinian, then it has finite length as a left module over itself.*

Proof. Consider the filtration $R = R_0 \supset R_1 \supset \cdots \supset R_s = 0$ where $R_i = (\operatorname{rad} R)^i$. Then each quotient R_i/R_{i+1} is semisimple of finite length. The statement follows. □

Let R be an artinian ring. By Krull-Schmidt theorem R (as a left module over itself) has a decomposition into direct sum of indecomposable submodules $R = L_1 \oplus \cdots \oplus L_n$. Since $\operatorname{End}_R(R) = R^{\mathrm{op}}$, the projector on each component L_i is given by multiplication on the right by some idempotent e_i. Thus, $R = Re_1 \oplus \cdots \oplus Re_n$, where e_i are idempotents and $e_ie_j = 0$ if $i \neq j$. This decomposition is unique up to multiplication on some unit on the right. Since Re_i is indecomposable, e_i can not be written as a sum of two orthogonal idempotents, such idempotents are called *primitive*. Each module Re_i is projective.

Lemma 4.8. *Let R be artinian, $N = \operatorname{rad} R$ and e be a primitive idempotent. Then Ne is a unique maximal submodule of Re.*

Proof. Since Re is indecomposable, every proper left ideal is nilpotent, (otherwise it has an idempotent and therefore splits as a direct summand in Re). But then this ideal is in $N \cap Re = Ne$. □

A projective module P is a *projective cover* of M if there exists a surjection $P \to M$.

Theorem 4.9. *Let R be artinian. Every simple R-modules S has a unique (up to an isomorphism) indecomposable projective cover isomorphic to Re for some primitive idempotent $e \in R$. Every indecomposable projective module has a unique (up to an isomorphism)simple quotient.*

Proof. Every simple S is a quotient of R, and therefore it is a quotient of some indecomposable projective $P = Re$. Let $\phi : P \to S$ be the natural projection. For any indecomposable projective cover P_1 of S with surjective morphism $\phi_1 : P_1 \to S$ there exist $f : P \to P_1$ and $g : P_1 \to P$ such that $\phi = \phi_1 \circ f$ and $\phi_1 = \phi \circ g$. Therefore $\phi = \phi \circ g \circ f$. Since $\operatorname{Ker} \phi$ is the unique maximal submodule, $g \circ f\,(P_1) = P$. In particular g is surjective. Indecomposability of P_1 implies $P \cong P_1$. Thus, every simple module has a unique indecomposable projective cover.

On the other hand, let P be an indecomposable projective module. Corollary 4.6 implies that P has a simple quotient S. Hence P is isomorphic to the indecomposable projective cover of S. □

Corollary 4.10. *Every indecomposable projective module over an artinian ring R is isomorphic to Re for some primitive idempotent $e \in R$. There is a bijection between the ismorphism classes of simple R-modules and isomorphism classes of projective indecomposable R-modules.*

Example. Let $R = \mathbb{F}_3(S_3)$. Let r be a 3-cycle and s be a transposition. Then $r - 1$, $r^2 - 1$, $sr - s$ and $sr^2 - s$ span a maximal nilpotent ideal. Hence R has two (up to an isomorphism) simple modules L_1 and L_2, where L_1 is a trivial representation of S_3 and L_2 is a sign representation. Choose primitive idempotents $e_1 = -s - 1$ and $e_2 = s - 1$, then $1 = e_1 + e_2$. Hence R has two indecomposable projective modules $P_1 = Re_1$ and $P_2 = Re_2$. Note that

$$Re_1 \cong \operatorname{Ind}_{S_2}^{S_3}(\mathrm{triv}),\ Re_2 \cong \operatorname{Ind}_{S_2}^{S_3}(\mathrm{sgn}).$$

Thus, P_1 is just 3-dimensional permutation representation of S_3, and P_2 is obtained from P_1 by tensoring with sgn. It is easy to see that P_1 has a trivial submodule as well as a trivial quotient, and sgn is isomorphic to the quotient of the maximal submodule of P_1 by the trivial submodule. One can easily get a similar description for P_2 Thus, one has the the following exact sequences

$$0 \to L_2 \to P_2 \to P_1 \to L_1 \to 0, \qquad 0 \to L_1 \to P_1 \to P_2 \to L_2 \to 0,$$

therefore

$$\cdots \to P_1 \to P_2 \to P_2 \to P_1 \to P_1 \to P_2 \to P_2 \to P_1 \to 0$$

is a projective resolution for L_1, and

$$\cdots \to P_2 \to P_1 \to P_1 \to P_2 \to P_2 \to P_1 \to P_1 \to P_2 \to 0$$

is a projective resolution for L_2. Now one can calculate Ext between simple modules

$$\operatorname{Ext}^k(L_i, L_i) = 0 \text{ if } k \equiv 1, 2 \mod 4, \qquad \operatorname{Ext}^k(L_i, L_i) = \mathbb{F}_3 \text{ if } k \equiv 0, 3 \mod 4,$$

and if $i \neq j$, then

$$\operatorname{Ext}^k(L_i, L_j) = 0 \text{ if } k \equiv 0, 3 \mod 4, \qquad \operatorname{Ext}^k(L_i, L_j) = \mathbb{F}_3 \text{ if } k \equiv 1, 2 \mod 4.$$

PROBLEM SET # 1
MATH 252

Due November 4.

1. Let R be the algebra of polynomial differential operators. In other words R is generated by x and $\frac{\partial}{\partial x}$ with relation

$$\frac{\partial}{\partial x}x - x\frac{\partial}{\partial x} = 1.$$

(The algebra R is called the Weyl algebra.) Let $M = \mathbb{C}[x]$ have a structure of R-module in the natural way. Show that $\mathrm{End}_R(M) = \mathbb{C}$, M is an iireducible R-module and the natural map $R \to \mathrm{End}_{\mathbb{C}}(M)$ is not surjective.

2. Let R be a subalgebra of upper triangular matrices in $\mathrm{Mat}_n(\mathbb{C})$. Classify simple and indecomposable projective modules over R and evaluate $\mathrm{Ext}^{\cdot}_R(M, N)$ for all simple M and N.

Date: October 27, 2005.

REPRESENTATION THEORY.
WEEKS 10 – 11

1. Representations of quivers

I follow here Crawley-Boevey lectures trying to give more details concerning extensions and exact sequences.

A *quiver* is an oriented graph. If Q is a quiver, then we denote by Q_0 the set of vertices and by Q_1 the set of arrows. Usually we denote by n the number of vertices. If $\gamma\colon j \leftarrow i$ is an arrow then $i = s(\gamma)$, $j = t(\gamma)$.

Fix an algebraically closed field k. A representation V of a quiver is a collection of vector spaces $\{V_i\}_{i\in Q_0}$ and linear maps $\rho_\gamma : V_i \to V_j$ for each arrow $\gamma\colon i \to j$. For two representations of a quiver Q, ρ in V and σ in W define a homomorphism $\phi\colon V \to W$ as a set of linear maps $\phi_i\colon V_i \to W_i$ such that the diagram

$$\begin{array}{ccc} V_j & \xleftarrow{\rho_\gamma} & V_i \\ \downarrow \phi_j & & \downarrow \phi_i \\ W_j & \xleftarrow{\sigma_\gamma} & W_i \end{array}$$

is commutative. We say that two representations V and W are isomorphic if there is a homomorphism $\phi \in \operatorname{Hom}_Q(V, W)$ such that each ϕ_i is an isomorphism. One can define a subrepresentation and a direct sum of representation of Q in the natural way. A representation is *irreducible* if it does not have non-trivial proper subrepresentation and *indecomposable* if it is not a direct sum of non-trivial subrepresentations.

Example 1.1. Let Q be the quiver $\bullet \to \bullet$. A representation of Q is a pair of vector spaces V and W and a linear operator $\rho : V \to W$. Let $V_0 = \operatorname{Ker}\rho$, V_1 is such that $V = V_0 \oplus V_1$, $W_0 = \operatorname{Im}\rho$, and W_1 is such that $W = W_0 \oplus W_1$. Then $V_0 \to 0$, $V_1 \to W_0$ and $0 \to W_1$ are subrepresentations and ρ is their direct sum. Furthermore, $V_0 \to 0$ is the direct sum of $\dim V_0$ copies of $k \to 0$, $V_1 \to W_0$ is the direct sum of $\dim V_1$ copies of $k \to k$ and finally $0 \to W_1$ is the direct sum of $\dim W_1$ copies of $0 \to k$. Thus, we see that there are exactly three isomorphism classes of indecomposable representations of Q, $0 \to k$, $k \to k$, $k \to 0$. The first and the last one are irreducible, $0 \to k$ is a subrepresentaion of $k \to k$ and $k \to 0$ is a quotient of $k \to k$ by $0 \to k$.

Date: December 4, 2005.

2. Path algebra

Given a quiver Q. A *path* p is a sequence $\gamma_1 \dots \gamma_k$ of arrows such that $s(\gamma_i) = t(\gamma_{i+1})$. Put $s(p) = s(\gamma_k)$, $t(p) = t(\gamma_1)$. Define a composition $p_1 p_2$ of two paths such that $s(p_1) = t(p_2)$ in the obvious way and we set $p_1 p_2 = 0$ if $s(p_1) \neq t(p_2)$. Introduce also elements e_i for each vertex $i \in Q_0$ and define $e_i e_j = \delta_{ij} e_i$, $e_i p = p$ if $i = t(p)$ and 0 otherwise, $p e_i = p$ if $i = s(p)$ and 0 otherwise. The *path algebra* $k(Q)$ is the set of k-linear combinations of all paths and e_i with composition extended by linearity from ones defined above.

One can easily check the following properties of a path algebra

1. $k(Q)$ is finite-dimensional iff Q does not have oriented cycles;
2. If Q is a disjoint union of Q_1 and Q_2, then $k(Q) = k(Q_1) \times k(Q_2)$;
3. The algebra $k(Q)$ has a natural $\mathbb{Z}$-grading $\oplus_{n=0}^{\infty} k(Q)_n$ defined by $\deg e_i = 0$ and the degree of a path p being the length of the path;
4. Elements e_i are primitive idempotents of $k(Q)$, and hence $k(Q) e_i$ is an indecomposable projective $k(Q)$-module.

The first three properties are trivial, let us check the last one. Suppose e_i is not primitive, then one can find an idempotent $\varepsilon \in k(Q) e_i$. Let $\varepsilon = c_0 e_i + c_1 p_1 + \dots + c_k p_k$, where $s(p_j) = i$ for all $j \leq k$. Then $\varepsilon^2 = \varepsilon$ implies $c_0 = 0$ or 1. Let $c_0 = 0$, $\varepsilon = \varepsilon_l + \dots$, where $\deg \varepsilon_l = l$ and other terms have degree greater than l. But then ε^2 starts with degree greater than $2l$, hence $\varepsilon = 0$. If $c_0 = 1$, apply the same argument to the idempotent $(e_i - \varepsilon)$.

Given a representation ρ of Q one can construct a $k(Q)$-module

$$V = \bigoplus_{i \in Q_0} V_i,\ e_i V_j = \delta_{ij} \operatorname{Id}_{V_j},\ \gamma v = \rho_\gamma v \text{ if } v \in V_{s(\gamma)},\ \gamma(v) = 0 \text{ otherwise.}$$

For any path $p = \gamma_1 \dots \gamma_k$ and $v \in V$ put $pv = \rho_{\gamma_1} \circ \dots \circ \rho_{\gamma_k}(v)$.

On the other hand, every $k(Q)$-module V defines a representation ρ of Q if one puts $V_i = e_i V$.

The following theorem is straightforward.

Theorem 2.1. *The category of representations of Q and the category of $k(Q)$-modules are equivalent.*

Lemma 2.2. *The radical of $k(Q)$ is spanned by all paths p satisfying the property that there is no return paths, i.e. back from $t(p)$ to $s(p)$.*

Proof. It is easy to see that the paths with no return span a two-sided ideal R. Note that $R^n = 0$, where n is the number of vertices. Thus, $R \subset \operatorname{rad} k(Q)$. On the other hand, let $y \notin R$ and p be a shortest path in decomposition of y which has a return path. Choose a shortest path s such that $\tau = sp$ is an oriented cycle. Consider the representation of Q which has k in each vertex of τ and 0 in all other vertices. Let $\rho_\gamma = \operatorname{Id}$, if γ is included in τ and $\rho_\gamma = 0$ otherwise. Let V be the corresponding $k(Q)$-module. Then V is simple, $sy(V) \neq 0$. Hence $y \notin \operatorname{rad} k(Q)$. Contradiction. □

Example 2.3. If Q has one vertex and n loops then $k(Q)$ is a free associative algebra with n generators. If Q does not have cycles, then $k(Q)$ is the subalgebra in $\mathrm{Mat}_n(k)$ generated by elementary matrices E_{ii} for each $i \in Q_0$ and E_{ij} for each arrow $i \to j$.

3. Standard resolution

Theorem 3.1. *Let Q be a quiver, $A = k(Q)$ and V be an A-module. Then the sequence*

$$0 \to \bigoplus_{\gamma=(i\to j)\in Q_1} Ae_j \otimes V_i \xrightarrow{f} \bigoplus_{i\in Q_0} Ae_i \otimes V_i \xrightarrow{g} V \to 0,$$

where $f(ae_j \otimes v) = ae_j\gamma \otimes v - ae_j \otimes \gamma v$, $g(ae_i \otimes v) = av$ for any $v \in V_i$, is exact. It is a projective resolution.

Proof. First, check that $g \circ f = 0$. Indeed,

$$g(f(ae_j \otimes v)) = g(ae_j\gamma \otimes v - ae_j \otimes \gamma v) = ae_j\gamma v - ae_j\gamma v = 0.$$

Since $V = \oplus e_i V_i$, g is surjective. To check that f is injective, introduce the grading on $A \otimes V$ using $\deg V = 0$. By grf denote the homogeneous part of highest degree for f. Note that the grf increases the degree by one and

$$grf = \oplus_{\gamma\in Q_1} f_\gamma, \text{ where } f_\gamma\colon Ae_j \otimes V_i \to A\gamma \otimes V_i \text{ is defined by}$$

$$f_\gamma(ae_j \otimes v_i) = ae_j\gamma \otimes v_i,$$

for $\gamma : i \to j$. One can see from this formula that f_γ is injective, therefore grf is injective and hence f is injective.

To prove that $\mathrm{Im}\, f = \mathrm{Ker}\, g$ note that

$$ae_j\gamma \otimes v \equiv ae_j \otimes \gamma v \mod \mathrm{Im}\, f,$$

therefore for any $x \in \bigoplus_{i\in Q_0} Ae_i \otimes V_i$

$$x \equiv x_0 \mod \mathrm{Im}\, f$$

for some x_0 of degree 0. In other words $x_0 \in \bigoplus_{i\in Q_0} ke_i \otimes V_i$. If $g(x) = 0$, then $g(x_0) = 0$, and if $g(x_0) = 0$, then obviously $x_0 = 0$. Hence $x \equiv 0 \mod \mathrm{Im}\, f$. □

Theorem 3.1 implies that $\mathrm{Ext}^1(X, Y)$ can be calculated as coker d of the following complex

$$0 \to \bigoplus_{i\in Q_0} \mathrm{Hom}_k(X_i, Y_i) \xrightarrow{d} \bigoplus_{\gamma=(i\to j)\in Q_1} \mathrm{Hom}_k(X_i, Y_j) \to 0, \tag{3.1}$$

where

$$d\phi(x) = \phi(\gamma x) - \gamma\phi(x) \tag{3.2}$$

for any $x \in X_i$, $\gamma = (i \to j)$.

Lemma 3.2. *Every* $\psi \in \operatorname{Ext}^1(X,Y)$ *induces a non-split exact sequence*

$$0 \to Y \to Z \to X \to 0.$$

If $\operatorname{Ext}^1(X,Y) = 0$, *then every exacts sequence as above splits.*

Proof. Let

$$0 \to Y \to Z \to X \to 0$$

be an exact sequence of representations of Q. Then Z_i can be identified with $X_i \oplus Y_i$ for every i. For every arrow $\gamma\colon i \to j$ the action on Z is defined by

$$\gamma(x,y) = (\gamma x, \gamma y + \psi_\gamma(x)),$$

for some $\psi_\gamma \in \operatorname{Hom}_k(X_i, Y_j)$. Thus, ψ can be considered as an element in the second non-zero term of (3.1). If the exact sequence splits, then there is $\eta \in \operatorname{Hom}_Q(X,Z)$ such that for each $i \in Q_0$, $x \in X_i$

$$\eta(x) = (x, \phi_i(x)),$$

for some $\phi_i \in \operatorname{Hom}_k(X_i, Y_i)$. Furthermore, $\eta \in \operatorname{Hom}_Q(X,Z)$ iff for each $\gamma\colon i \to j$

$$\gamma(x, \phi_i(x)) = (\gamma x, \gamma\phi_i(x) + \psi_\gamma(x)) = (\gamma x, \phi_j(\gamma x)),$$

which implies

$$\psi_\gamma(x) = \phi_j(\gamma x) - \gamma\phi_i(x).$$

In other words, $\psi = d\phi$. Thus, $\operatorname{Ext}^1(X,Y)$ parameterizes the set of non-split exact sequences

$$0 \to Y \to Z \to X \to 0.$$

□

Corollary 3.3. *In the category of representations of* Q, $\operatorname{Ext}^i(X,Y) = 0$ *for* $i \geq 2$.

Corollary 3.4. *Let*

$$0 \to Y \to Z \to X \to 0$$

be a short exact sequence of representations of Q, *then*

$$\operatorname{Ext}^1(V,Z) \to \operatorname{Ext}^1(V,X),\ \operatorname{Ext}^1(Z,V) \to \operatorname{Ext}^1(Y,V)$$

are surjective.

Lemma 3.5. *If* X *and* Y *are indecomposable and* $\operatorname{Ext}^1(Y,X) = 0$, *then every non-zero* $\varphi \in \operatorname{Hom}_Q(X,Y)$ *is either surjective or injective.*

Proof. Use the exact sequences

$$0 \to \operatorname{Ker}\varphi \to X \to \operatorname{Im}\varphi \to 0,$$

$$0 \to \operatorname{Im}\varphi \to Y \to S \cong Y/\operatorname{Im}\varphi \to 0. \tag{3.3}$$

The exact sequence (3.3) can be considered as an element $\psi \in \operatorname{Ext}^1(S, \operatorname{Im}\varphi)$ by use of Lemma 3.2. By Corollary 3.3 we have an isomorphism $g\colon \operatorname{Ext}^1(S, \operatorname{Im}\varphi) \cong \operatorname{Ext}^1(S, X)$. Then $g(\psi)$ induces the exact sequence

$$0 \to X \to Z \to S \to 0,$$

and this exact sequence together with (3.3) form the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & X & \stackrel{\alpha}{\to} & Z & \to & S & \to & 0 \\ & & \downarrow^{\beta} & & \downarrow^{\gamma} & & \downarrow & & \\ 0 & \to & \operatorname{Im}\varphi & \stackrel{\delta}{\to} & Y & \to & S & \to & 0 \end{array}$$

here β and γ are surjective. We claim that the sequence

$$0 \to X \xrightarrow{\alpha+\beta} Z \oplus \operatorname{Im}\varphi \xrightarrow{\gamma-\delta} Y \to 0$$

is exact. Indeed, $\alpha+\beta$ is obviously injective and $\gamma-\delta$ is surjecitve. Finally, $\dim Z = \dim X + \dim S$, $\dim \operatorname{Im}\varphi = \dim Y - \dim S$. Therefore,

$$\dim(Z \oplus \operatorname{Im}\varphi) = \dim X + \dim Y,$$

and therefore $\operatorname{Ker}(\gamma-\delta) = \operatorname{Im}(\alpha+\beta)$.

But $\operatorname{Ext}^1(Y, X) = 0$. Hence the last exact sequence splits, $Z \oplus \operatorname{Im}\varphi \cong X \oplus \mathrm{Yand}$ by Krull-Schmidt theorem either $X \cong \operatorname{Im}\varphi$ or $Y \cong \operatorname{Im}\varphi$. □

Intoduce $\dim X$ as a vector $x = (x_1, \dots, x_n) \in \mathbb{Z}^n$ where n is the number of vertices and $x_i = \dim X_i$. Define the bilinear form

$$\langle x, y\rangle = \sum_{i\in Q_0} x_i y_i - \sum_{(i\to j)\in Q_1} x_i y_j = \dim \operatorname{Hom}_Q(X, Y) - \dim \operatorname{Ext}^1(X, Y)$$

(the equality follows from (3.1)). We also introduce the symmetric form

$$(x, y) = \langle x, y\rangle + \langle y, x\rangle$$

and the quadratic form

$$q(x) = \langle x, x\rangle .$$

4. Bricks

Here we discuss further properties of finite-dimensional representations of $A = k(Q)$.

Recall that if X is indecomposable and has finite length, then $\varphi \in \operatorname{End}_Q(X)$ is either isomorphism or nilpotent. Since we assumed that k is algebraically closed, $\varphi = \lambda \operatorname{Id}$ for any invertible $\varphi \in \operatorname{End}_Q(X)$. A representation X is a *brick*, if $\operatorname{End}_Q(X) = k$. If X is a brick, then X is indecomposable. If X is indecomposable and $\operatorname{Ext}^1(X, X) = 0$, then X is a brick due to Lemma 3.5.

Example 4.1. Consider the quiver $\bullet \to \bullet$. Then every indecomposable is a brick. For the Kronecker quiver $\bullet \rightrightarrows \bullet$ the representation $k^2 \rightrightarrows_{\beta}^{\alpha} k^2$ with $\alpha = \mathrm{Id}$, $\beta = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ is not a brick. Indeed, $\varphi = (\varphi_1, \varphi_2)$ where φ_1, φ_2 are matrices $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, belongs to $\mathrm{End}_Q(X)$.

Lemma 4.2. *Let X be indecomposable and not a brick, then X contains a brick W such that* $\mathrm{Ext}^1(W, W) \neq 0$.

Proof. Choose $\varphi \in \mathrm{End}_Q(X)$, $\varphi \neq 0$ of minimal rank. Since $\mathrm{rk}\,\varphi^2 < \mathrm{rk}\,\varphi$, $\varphi^2 = 0$. Let $Y = \mathrm{Im}\,\varphi$, $Z = \mathrm{Ker}\,\varphi$. Let $Z = Z_1 \oplus \cdots \oplus Z_p$ be a sum of indecomposables. Let $p_i : Z \to Z_i$ be the projection. Choose i so that $p_i(Y) \neq 0$ and let $\eta = p_i \circ \varphi \in \mathrm{End}_Q(X)$ (well defined since $\mathrm{Im}\,\varphi \in \mathrm{Ker}\,\varphi$). Note that by our assumption $\mathrm{rk}\,\eta = \mathrm{rk}\,\varphi$, therefore $p_i : Y \to Z_i$ is an embedding. Let $Y_i = p_i(Y)$. Then $\mathrm{Ker}\,\eta = Z$, $\mathrm{Im}\,\eta = Y_i$.

We claim now that $\mathrm{Ext}^1(Z_i, Z_i) \neq 0$. Indeed, $\mathrm{Ext}^1(Y_i, Z) \neq 0$ by exact sequence

$$0 \to Z \to X \xrightarrow{\eta} Y_i \to 0$$

and indecomposability of X. Then the induced exact sequence

$$0 \to Z_i \to X_i \xrightarrow{\eta} Y_i \to 0$$

does not split also. (If it splits, then Z_i is a direct summand of X, which is impossible). Therefore $\mathrm{Ext}^1(Y_i, Z_i) \neq 0$. But Y_i is a submodule of Z_i. By Corollary 3.4 we have the surjection

$$\mathrm{Ext}^1(Z_i, Z_i) \to \mathrm{Ext}^1(Y_i, Z_i).$$

If Z_i is not a brick, we repeat the above construction for Z_i e.t.c. Finally, we get a brick. □

Corollary 4.3. *Assume that the quadratic form q is positive definite. Then every indecomposable X is a brick with trivial* $\mathrm{Ext}^1(X, X)$*; moreover, if $x = \dim X$, then* $q(x) = 1$.

Proof. Assume that X is not a brick, then it contains a brick Y such that $\mathrm{Ext}^1(Y, Y) \neq 0$. Then

$$q(Y) = \dim \mathrm{End}_Q(Y) - \dim \mathrm{Ext}^1(Y, Y) = 1 - \dim \mathrm{Ext}^1(Y, Y) \leq 0,$$

but this is impossible. Therefore X is a brick. Now

$$q(x) = \dim \mathrm{End}_Q(X) - \dim \mathrm{Ext}^1(X, X) = 1 - \dim \mathrm{Ext}^1(X, X) \geq 0,$$

hence $q(x) = 1$ and $\dim \mathrm{Ext}^1(X, X) = 0$. □

5. Orbits in representation variety

Fix a quiver Q, recall that n denotes the number of vertices. Let $x = (x_1, \ldots, x_n) \in \mathbb{Z}^n_{\geq 0}$. Define

$$\mathrm{Rep}(x) = \prod_{(i \to j) \in Q_1} \mathrm{Hom}_k\left(k^{x_i}, k^{x_j}\right).$$

It is clear that every representation of Q of dimension x is a point in $\mathrm{Rep}\,(x)$. Let

$$G = \prod_{i \in Q_0} \mathrm{GL}\left(k^i\right).$$

Then G acts on $\mathrm{Rep}\,(x)$ by the formula $g\varphi_{ij} = g_i \varphi g_j^{-1}$, for each arrow $i \to j$. Two representations of Q are isomorphic iff they belong to the same orbit of G. For a representation X we denote by O_X the corresponding G-orbit in $\mathrm{Rep}\,(x)$.

Note that

$$\dim \mathrm{Rep}\,(x) = \sum_{(i \to j) \in Q_1} x_i x_j, \; \dim G = \sum_{i \in Q_0} x_i^2,$$

therefore

$$\dim \mathrm{Rep}\,(x) - \dim G = -q\,(x)\,. \tag{5.1}$$

Since G is an affine algebraic group acting on an affine algebraic variety, we can work in Zariski topology. Then each orbit is open in its closure, if O and O' are two orbits and $O' \subset \bar{O}$, $O \neq O'$, then $\dim O' < \dim O$. Finally, we need the formula

$$\dim O_X = \dim G - \dim \mathrm{Stab}_X,$$

here Stab_X stands for the stabilizer of X. Also note that in our case the group G is connected, therefore each G-orbit is irreducible.

Lemma 5.1. $\dim \mathrm{Stab}_X = \dim \mathrm{Aut}_Q\,(X) = \dim \mathrm{End}_Q\,(X)$.

Proof. The condition that $\phi \in \mathrm{End}_Q\,(X)$ is not invertible is given by the polynomial equations $\det \phi_i = 0$. Since $\mathrm{Aut}_Q\,(X)$ is not empty, we are done. □

Corollary 5.2.

$$\operatorname{codim} O_X = \dim \mathrm{Rep}\,(x) - \dim G + \dim \mathrm{Stab}_X = -q\,(x) + \dim \mathrm{End}_Q\,(X) = \dim \mathrm{Ext}^1\,(X, X)\,.$$

Lemma 5.3. *Let Z be a nontrivial extension of Y by X, i.e. there is a non-split exact sequence*

$$0 \to X \to Z \to Y \to 0.$$

Then $O_{X \oplus Y} \subset \bar{O}_Z$ and $O_{X \oplus Y} \neq O_Z$.

Proof. Write each Z_i as $X_i \oplus Y_i$ and define $g_i^{\lambda}|_{X_i} = \mathrm{Id}$, $g_i^{\lambda}|_{Y_i} = \lambda \mathrm{Id}$ for any $\lambda \neq 0$. Then obviously $X \oplus Y$ belongs to the closure of $g_i^{\lambda}\,(Z)$. It is left to check that $X \oplus Y \not\cong Z$. But the sequence is non-split, therefore

$$\dim \mathrm{Hom}_Q\,(Y, Z) < \dim \mathrm{Hom}_Q\,(Y, X \oplus Y)\,.$$

□

Corollary 5.4. *If O_X is closed then X is semisimple.*

6. Dynkin and affine graphs

Let Γ be a connected graph with n vertices, then Γ defines a symmetric bilinear form $(\cdot,\cdot)$ on $\mathbb{Z}^n$

$$(x,y)=\sum_{i\in\Gamma_0}2x_iy_i-\sum_{(i,j)\in\Gamma_1}x_iy_j.$$

If Γ is equipped with orientation then the symmetric form coincides with the introduced earlier symmetric form of the corresponding quiver. The matrix of the form $(\cdot,\cdot)$ in the standard basis is called the *Cartan matrix* of Γ.

Example 6.1. The Cartan matrix of $\bullet-\bullet$ is $\left(\begin{smallmatrix}2 & -1\\ -1 & 2\end{smallmatrix}\right)$.

Theorem 6.2. *Given a connected graph Γ, exactly one of the following conditions holds*

1. *The symmetric $(\cdot,\cdot)$form is positive definite, then Γ is called Dynkin graph.*
2. *The symmetric form $(\cdot,\cdot)$is positive semidefinite, there exist $\delta\in\mathbb{Z}^n_{>0}$ such that $(\delta,x)=0$ for any $x\in\mathbb{Z}^n$. The kernel of $(\cdot,\cdot)$ is $\mathbb{Z}\delta$. In this case Γ is called affine or Euclidean.*
3. *There is $x\in\mathbb{Z}^n_{\geq 0}$ such that $(x,x)<0$. Then Γ is called of indefinite type.*

A Dynkin graphs is one of A_n, D_n, E_6, E_7, E_8. An affine graphs is one of $\hat{A}_n, \widehat{D}_n, \widehat{E}_6, \widehat{E}_7, \widehat{E}_8$. Every affine graph is obtained from a Dynkin graph by adding one vertex.

Proof. First, we check that A_n, D_n, E_6, E_7, E_8 define a positive definite form using the Sylvester criterion and the fact that every subgraph of a Dynkin graph is Dynkin. One can calculate the determinant of the Cartan matrix inductively. It is $n+1$ for A_n, 4 for D_n, 3 for E_6, 2 for E_7 and 1 for E_8. In the same way one can check that the Cartan matrices of affine graphs have determinant 0 and corank 1. The rows are linearly dependent with positive coefficients. Any other graph Γ has an affine graph Γ' as a subgraph, hence either $(\delta,\delta)<0$ or $(2\delta+\alpha_i,2\delta+\alpha_i)<0$, if α_i is the basis vector corresponding to a vertex i which does not belong to Γ' but is connected to some vertex of Γ'. □

A vector $\alpha\in\mathbb{Z}^n$ is called a *root* if $q(\alpha)=\frac{(\alpha,\alpha)}{2}\leq 1$. It is clear that $\alpha_1,\dots,\alpha_n$ are roots. They are called *simple roots*.

Lemma 6.3. *Let Γ be Dynkin or affine. If α is a root and $\alpha=m_1\alpha_1+\dots+m_n\alpha_n$, then either all $m_i\geq 0$ or all $m_i\leq 0$.*

Proof. Let $\alpha=\beta-\gamma$, where $\beta=\sum_{i\in I}m_i\alpha_i$, $\gamma=\sum_{j\notin I}m_j\alpha_j$ for some $m_i,m_j\geq 0$, then $q(\alpha)=q(\beta)+q(\gamma)-(\beta,\gamma)$. Since Γ is Dynkin or affine, then $q(\beta)\geq 0$, $q(\gamma)\geq 0$. On the other hand $(\beta,\gamma)\leq 0$. Since $q(\alpha)\leq 1$, only one of three terms $q(\beta)$, $q(\gamma)$, $-(\beta,\gamma)$ can be positive, which is possible only if β or γ is zero. □

A root α is positive if $\alpha=m_1\alpha_1+\dots+m_n\alpha_n$, $m_i\geq 0$ for all i.

A quiver has *finite type* if there are finitely many isomorphism classes of indecomposable representations.

Theorem 6.4. *(Gabriel) A connected quiver Q has finite type iff the corresponding graph is Dynkin. For a Dynkin quiver there exists a bijection between positive roots and isomorphism classes of indecomposable representations.*

Proof. If Q is of finite type, then $\operatorname{Rep}(x)$ has finitely many orbits for each $x \in \mathbb{Z}^n_{\geq 0}$. If Q is not Dynkin, then there exists $x \in \mathbb{Z}^n_{\geq 0}$ such that $q(x) \leq 0$. If Q has finite type, then $\operatorname{Rep}(x)$ must have an open orbit O_X. By Corollary 5.2

$$\operatorname{codim} O_X = \dim \operatorname{End}_Q(X) - q(x) > 0. \tag{6.1}$$

Contradiction.

Now suppose that Q is Dynkin. Every indecomposable representation X is a brick with trivial self-extensions by Corollary 4.3. Hence $q(x) = 1$, i.e. x is a root. By (6.1) O_X is the unique open orbit in $\operatorname{Rep}(x)$. What remains is to show that for each root x there exists an indecomposable representation of dimension x. Indeed, let X be such that $\dim O_X$ in $\operatorname{Rep}(x)$ is maximal. We claim that X is indecomposable. Indeed, let $X = X_1 \oplus \cdots \oplus X_s$ be a sum of indecomposable bricks. Then by Lemma 5.3 $\operatorname{Ext}^1(X_i, X_j) = 0$. Therefore $q(x) = s = 1$. Hence X is indecomposable. □

PROBLEM SET # 10
MATH 252

Due November 14.

1. Let Q be a connected quiver and $k(Q)$ be the path algebra of Q. Show that the center of $k(Q)$ is isomorphic either to k, or to $k[x]$, and that the latter happens only in the case when Q is an oriented cycle.

2. Classify indecomposable representations of the quiver:

$$\bullet \to \bullet \leftarrow \bullet.$$

3. Classify indecomposable representations of the quiver:

$$\begin{array}{ccccc} \bullet & \to & \bullet & \leftarrow & \bullet \\ & & \uparrow & & \\ & & \bullet & & \end{array}$$

Date: November 4, 2005.

REPRESENTATION THEORY
WEEK 12

1. Reflection functors

Let Q be a quiver. We say a vertex $i \in Q_0$ is $+$-*admissible* if all arrows containing i have i as a target. If all arrows containing i have i as a source, we call i $-$-*admissible.* By $\sigma_i(Q)$ we denote the quiver obtained from Q by inverting all arrows containing i.

Let i be a $+$-admissible vertex and $Q' = \sigma_i(Q)$. Let us introduce the functor F_i^+: $\mathrm{Rep}_Q \to \mathrm{Rep}_{Q'}$. Let X be a representation of Q. Define $X' = F_i^+ X$ as follows. If $j \neq i$, then $X'_j = X_j$. Put $X'_i = \operatorname{Ker} h$, where

$$h = \sum_{\gamma=(j\to i)\in Q_1} \rho_\gamma : \bigoplus X_j \to X_i,$$

For each $\gamma = (i \to j) \in Q'$ define $\rho'_\gamma : X'_i \to X_j = X'_j$ as the natural projection on the component $X_j \in \bigoplus X_j$.

If i is a $-$-admissible vertex and $Q' = \sigma_i(Q)$ one can define the functor F_i^-: $\mathrm{Rep}_Q \to \mathrm{Rep}_{Q'}$ as follows. Let $X' = F_i^-(X)$, where $X'_j = X_j$ for $i \neq j$, and $X'_i = \operatorname{Coker} \widetilde{h}$, where

$$\widetilde{h} = \sum_{\gamma=(i\to j)\in Q_1} \rho_\gamma : X_i \to \oplus X_j,$$

and for each $\gamma = (j \to i) \in Q'$ define $\rho'_\gamma \colon X_j = X'_j \to X'_i$ by restriction of the projection $\oplus X_j \to \operatorname{Coker} \widetilde{h}$ to X_j.

Example. Let Q be the quiver $1 \to 2$, and X is the representation $k \to 0$, then $F_1^-(X) = 0$ and $F_2^+(X)$ is $k \leftarrow k$.

It is easy to check that F_i^+ is left-exact (maps an injection to an injection) and F_i^- is right exact (maps a surjection to a surjection). Let L_i denote the representation of Q which has k in the vertex i and zero in all other vertices. Then $F_i^+(L_i) = 0$ and $F_i^-(L_i) = 0$.

Theorem 1.1. *Let X be an indecomposable representation of Q and i be a $+$-admissible vertex. Then $F_i^+(X) = 0$ iff $X \cong L_i$. Otherwise $X' = F_i^+(X)$ is indecomposable,*

$$\dim X'_i = -\dim X_i + \sum_{(j\to i)} \dim X_j \tag{1.1}$$

and $F_i^- F_i^+(X) \cong X$.

Date: November 27, 2005.

If i is $-$-admissible vertex and X is indecomposable, then $F_i^-(X)=0$ iff $X\cong L_i$. Otherwise $F_i^-(X)$ is indecomposable, the dimension of $F_i^-(X)$ can be calculated by the same formula (1.1) *and $F_i^+F_i^-(X)\cong X$.*

Proof. Note that if $X\ncong L_i$, then h must be surjective because of indecomposability of X, hence the formula (1.1) holds. Furthermore, we have the following exact sequence

$$0\to X_i'\xrightarrow{\widetilde{h}}\bigoplus_{(j\to i)\in Q_1}X_j\xrightarrow{h}X_i\to 0 \tag{1.2}$$

and $\left(F_i^-X'\right)_i=\operatorname{Coker}\widetilde{h}\cong X_i$. Observe also that $\widetilde{h}$ is injective by definition, hence X' is indecomposable. Thus, we proved the first part of the theorem.

For the case of $-$-admissible vertex, define D: Rep $Q\to$ Rep Q^{op}, where Q^{op} is the quiver with all arrows of Q reversed, $D(X_j)=X_j^*$, $D(\rho_\gamma)=\rho_\gamma^*$, and note that $D\circ F_i^+=F_i^-\circ D$. Since D reverses all maps and change Ker to Coker, the second statement of the theorem follows immediately. □

Note that for an arbitrary X the sequence (1.2) is not exact but $\widetilde{h}$ is injective and $h\circ\widetilde{h}=0$. Therefore one can define a natural injection $\phi:F_i^-F_i^+X\to X$, where $\phi_j=\mathrm{id}$ for all $j\neq i$ and ϕ_i coincides with h restricted to $\operatorname{Coker}\widetilde{h}$. In the similar way one can define the natural surjection $\psi:X\to F_i^+F_i^-X$ if i is a $-$-admissible vertex.

Finally let $Q'=\sigma_i(Q)$, X be a representation of Q' and Y be a representation of Q, $X'=F_i^-X$ and $Y'=F_i^+Y$. Let $\eta\in\operatorname{Hom}_Q(X,Y')$, define $\chi\in\operatorname{Hom}_{Q'}(X',Y)$ by putting $\chi_j=\mathrm{Id}$ for $j\neq i$ and obtaining χ_i from following commutative diagram

$$\begin{array}{ccccccc} & X_i & \xrightarrow{\widetilde{h}} & \oplus X_j & \xrightarrow{h} & X_i' & \to & 0\\ & \downarrow^{\eta_i} & & \downarrow^{\oplus\eta_j} & & \downarrow^{\chi_i} & & \\ 0 & \to Y_i' & \xrightarrow{\widetilde{h}} & \oplus Y_j' & \xrightarrow{h} & Y_i & & \end{array}$$

Note χ_i is uniquely determined by η. In the same way for each $\chi\in\operatorname{Hom}_Q(X',Y)$ one can define $\eta\in\operatorname{Hom}_{Q'}(X,Y')$. A routine check now proves the following

Lemma 1.2. *Let $Q'=\sigma_i(Q)$, X be a representation of Q' and Y be a representation of Q, then*

$$\operatorname{Hom}_{Q'}\left(F_i^-X,Y\right)\cong\operatorname{Hom}_Q\left(X,F_i^+Y\right).$$

2. Reflection functors and change of orientation.

Lemma 2.1. *Let Γ be a connected graph without cycles, Q and Q' be two quivers on the same graph. Then there exists an enumeration of vertices such that $Q'=\sigma_k\circ\cdots\circ\sigma_1(Q)$ and i is a $+$-admissible vertex for $\sigma_{i-1}\circ\cdots\circ\sigma_1(Q)$.*

Proof. It is sufficient to prove the statement for two quivers Q and Q' different at one arrow. So let $\gamma \in Q_1$. After removing γ, Q splits in two connected components; let Q'' be the component which contains $t(\gamma)$. Enumerate vertices of Q'' in such a way that if $i \to j \in Q''_1$, then $i > j$. This is possible since Q'' does not have cycles. Check that $Q' = \sigma_k \circ \dots \circ \sigma_1(Q)$ (here k is the number of all vertices in Q'') and i is a $+$-admissible vertex for $\sigma_{i-1} \circ \dots \circ \sigma_1(Q)$. □

Theorem 2.2. *Let i be a $+$-admissible vertex for Q and $Q' = \sigma_i(Q)$. Then F_i^+ and F_i^- establish a bijection between indecomposable representations of Q (non-isomorphic to L_i) and indecomposable representations of Q'(non-isomorphic to L_i.)*[1]

Theorem 2.2 follows from 1.1. Together with Lemma 2.1 it allows to change an orientation on a quiver if the quiver does not have cycles.

3. Weyl group and reflection functors.

Given any graph Γ, one can associate with it a certain linear group, which is called a Weyl group of Γ. We denote by $\alpha_1, \dots, \alpha_n$ vectors in the standard basis of $\mathbb{Z}^{\Gamma_0} = \mathbb{Z}^n$, α_i corresponds to the vertex i. These vectors are called simple roots. For each simple root α_i put

$$r_i(x) = x - \frac{2(x, \alpha_i)}{(\alpha_i, \alpha_i)}\alpha_i.$$

One can check that r_i preserves the scalar product and $r_i^2 = id$. The linear transformation r_i is called a *simple reflection.* If Γ has no loops, r_i also preserves the lattice generated by simple roots. Hence r_i maps roots to roots. If Γ is Dynkin, the scalar product is positive-definite, and r_i is a reflection in the hyperplane orthogonal to α_i. The *Weyl group* W is a group generated by $r_1, \dots, r_n$. For a Dynkin diagram W is finite (since the number of roots is finite).

Example. Let $\Gamma = A_n$. Let $\varepsilon_1, \dots, \varepsilon_{n+1}$ be an orthonormal basis in $\mathbb{R}^{n+1}$. Then one can take the roots of Γ to be $\varepsilon_i - \varepsilon_j$, simple roots to be $\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \dots, \varepsilon_n - \varepsilon_{n+1}$, $r_i(\varepsilon_j) = 0$ if $j \neq i, i+1$, and $r_i(\varepsilon_i) = \varepsilon_{i+1}$. Therefore W is isomorphic to the permutation group S_{n+1}.

One can check by direct calculation, that (1.1) implies

Lemma 3.1. *If X is an indecomposable representation of Q and $\dim X = x \neq \alpha_i$, then $\dim F_i^{\pm} X = r_i(x)$.*

An element $c = r_1 \dots r_n \in W$ is called a *Coxeter transformation.* It depends on the enumeration of simple roots.

Example. In the case $\Gamma = A_n$ a Coxeter element is always a cycle of length $n+1$.

Lemma 3.2. *If $c(x) = x$, then $(x, \alpha_i) = 0$ for all i. In particular for a Dynkin graph $c(x) = x$ implies $x = 0$.*

[1] We will denote by the same letter L_i the representations of quivers with different orientation.

Proof. By definition,

$$c(x) = x + a_1\alpha_1 + \dots + a_n\alpha_n,\ a_i = -\frac{2(\alpha_i, x + a_1\alpha_1 + \dots a_{i-1}\alpha_{i-1})}{(\alpha_i, \alpha_i)}.$$

The condition $c(x) = x$ implies all $a_i = 0$. Hence $(x, \alpha_i) = 0$ for all i. □

4. Coxeter functor.

Let Q be a graph without oriented cycles. We call an enumeration of vertices admissible if $i > j$ for any arrow $i \to j$. Such an enumeration always exists. One can easily see that every vertex i is a $+$-admissible for $\sigma_{i-1} \circ \dots \circ \sigma_1(Q)$ and $-$-admissible for $\sigma_{i+1} \circ \dots \circ \sigma_n(Q)$. Furthermore,

$$Q = \sigma_n \circ \sigma_{n-1} \circ \dots \circ \sigma_1(Q) = \sigma_1 \circ \dots \circ \sigma_n(Q).$$

Define Coxeter functors

$$\Phi^+ = F_n^+ \circ \dots \circ F_2^+ \circ F_1^+,\ \Phi^- = F_1^- \circ F_2^- \circ \dots \circ F_n^-.$$

Lemma 4.1. (1) *$Hom_Q(\Phi^- X, Y) \cong \mathrm{Hom}_Q(X, \Phi^+ Y)$;*
(2) *If X is indecomposable and $\Phi^+ X \neq 0$, then $\Phi^-\Phi^+ X \cong X$;*
(3) *If X is indecomposable of dimension x and $\Phi^+ X \neq 0$, then $\dim \Phi^+ X = c(x)$;*
(4) *If Q is Dynkin, then for any indecomposable X there exists k such that $(\Phi^+)^k X = 0$.*

Proof. (1) follows from Lemma 1.2, (2) follows from Theorem 1.1, (3) follows from Lemma 3.1. Let us prove (4). Since W is finite, c has finite order h. It is sufficient to show that for any x there exists k such that $c^k(x)$ is not positive. Assume that this is not true. Then $y = x + c(x) + \dots + c^{h-1}(x) > 0$ is c invariant. Contradiction with Lemma 3.2.

□

Lemma 4.2. $\Phi^\pm$ *does not depend on a choice of admissible enumeration.*

Proof. Note that if i and j are disjoint and both $+(-)$-admissible, then $F_i^+ \circ F_j^+ = F_j^+ \circ F_i^+ \left(F_i^- \circ F_j^- = F_j^- \circ F_i^-\right)$. If a sequence $i_1, \dots, i_n$ gives another admissible enumeration of vertices, and $i_k = 1$, then 1 is disjoint with $i_1, \dots, i_{k-1}$, hence

$$F_1^+ \circ F_{i_{k-1}}^+ \circ \dots \circ F_{i_1}^+ = F_{i_{k-1}}^+ \circ \dots \circ F_{i_1}^+ \circ F_1^+.$$

Now proceed by induction. Similarly for Φ^-. □

In what follows we always assume that an enumeration of vertices is admissible.

Corollary 4.3. *Let Q be a Dynkin quiver, X be an indecomposable representation of dimension x, and k be the minimal number such that $c^{k+1}(x)$ is not positive. There exists a unique vertex i such that*

$$x = c^{-k} r_1 \dots r_{i-1}(\alpha_i),\ X \cong \left(\Phi^-\right)^k \circ F_1^- \circ \dots \circ F_{i-1}^-(L_i).$$

In particular, x is a positive root and for each positive root x, there is a unique (up to an isomorphism) indecomposable representation of dimension x.

Proof. Follows from Theorem 1.1 and Lemma 3.1. □

5. Further properties of Coxeter functors

Here we assume again that Q is a quiver without oriented cycles and the enumeration of vertices is admissible. We discuss the properties of the bilinear form $\langle , \rangle$. Since we plan to change an orientation of Q we use a subindex $\langle , \rangle_Q$, where it is needed to avoid ambiguity.

Lemma 5.1. *Let i be a $+$-admissible vertex, $Q' = \sigma_i(Q)$, and $\langle , \rangle_Q$, $\langle , \rangle_{Q'}$ the corresponding bilinear forms. Then*

$$\langle r_i(x), y\rangle_{Q'} = \langle x, r_i(y)\rangle_Q .$$

Proof. It suffices to check the formula for a subquiver containing i and all its neighbors. Let $x' = r_i(x)$ and $y' = r_i(y)$. Then

$$x'_i = -x_i + \sum_{i\neq j} x_j,\; y'_i = -y_i + \sum_{i\neq j} y_j,$$

$$\langle x', y\rangle_{Q'} = x'_i y_i - x'_i \sum_{i\neq j} y_j + \sum_{i\neq j} x_j y_j = -x'_i y'_i + \sum_{i\neq j} x_j y_j,$$

$$\langle x, y'\rangle_Q = x_i y'_i - y'_i \sum_{i\neq j} x_j = -x'_i y'_i + \sum_{i\neq j} x_j y_j.$$

□

Corollary 5.2. *For a Coxeter element c we have*

$$\left\langle c^{-1}(x), y\right\rangle = \langle x, c(y)\rangle .$$

If $\Phi^+(Y) \neq 0, \Phi^-(X) \neq 0$, then

$$\dim \operatorname{Ext}^1\left(X, \Phi^+(Y)\right) = \dim \operatorname{Ext}^1\left(\Phi^-(X), Y\right).$$

Proof. First statement follows directly from Lemma 5.1. The second statement follows from the first statement, Lemma 1.2 and the identity

$$\langle x, y\rangle_Q = \dim \operatorname{Hom}_Q(X, Y) - \dim \operatorname{Ext}^1(X, Y).$$

□

Let $A = k(Q)$ be the path algebra. Recall that any indecomposable projective module is isomorphic to Ae_i.

Lemma 5.3. $F_i^+ \circ \cdots \circ F_1^+(Ae_i) = 0$, $F_{i-1}^+ \circ \cdots \circ F_1^+(Ae_i) \cong L_i$.

Proof. One can check by direct calculation that for each component e_jAe_i, $e_jAe_i = 0$ for $j > i$, and

$$F_k^+ \circ \dots \circ F_1^+ (e_jAe_i) = e_jAe_i \text{ for } k < j,\ F_j^+ \circ \dots \circ F_1^+ (e_jAe_i) = 0.$$

□

Corollary 5.4. *$\Phi^+ (P) = 0$ for any projective module P. For any indecomposable projective Ae_i we have*

$$Ae_i = F_1^- \circ \dots \circ F_{i-1}^- (L_i) . \tag{5.1}$$

Proof. The first statement follows from Lemma 5.3 immediately. For the second use Theorem 1.1 and Lemma 5.3.

□

An injective module is a module I such that for any injective homomorphism $i : X \to Y$ and any homomorphism $\varphi : X \to I$, there exists a homomorphism $\psi : Y \to I$ such that $\varphi = \psi \circ i$. A module I is injective iff $\mathrm{Ext}^1 (X, I) = 0$ for any X. One can see analogy with projective modules, however in general there is no nice description of injective (like a summand of a free module).

Exercise. Check that $\mathbb{Q}$ is an injective $\mathbb{Z}$-module.

In case when A is a finite-dimensional algebra, injective modules are easy to describe. Indeed, the functor $D : A-\ \text{mod} \to \text{mod}\ -A$ such that $D (X) = X^*$ maps left projective modules to right injective and vice versa. Therefore any indecomposable injective module is isomorphic to $(e_jA)^*$. Since $D \circ \Phi^+ = \Phi^- \circ D$, one can see easily that $\Phi^- (I) = 0$ for any injective module I. Moreover, one can prove similarly to the projective case that

$$(e_jA)^* \cong F_n^+ \circ \dots \circ F_{j+1}^+ (L_j) .$$

Let $P (j) = Ae_j$ and $I (j) = (e_jA)^*$ and $p (j) = \dim P (j)$, $i (j) = \dim I (j)$. Then

$$c (p (j)) = r_n \dots r_1 (p (j)) = r_n \dots r_{j+1} (-\alpha_j) = -i (j) . \tag{5.2}$$

Note that $\mathrm{Ext}^1 (Ae_j, X) = 0$ for any X and $\dim \mathrm{Hom}_Q (Ae_j, X) = x_j$. Hence

$$\langle p (j) , x\rangle = x_j. \tag{5.3}$$

On the other hand, $\mathrm{Ext}^1 (X, (e_jA)^*) = 0$ and

$$\mathrm{Hom}_Q (X, (e_jA)^*) \cong \mathrm{Hom}_Q (e_jA, X^*) ,$$

which implies $\dim \mathrm{Hom}_Q (X, (e_jA)^*) = x_j$. Thus, we obtain

$$\langle x, i (j)\rangle = x_j. \tag{5.4}$$

Combine together (5.2), (5.3), (5.4) and get

$$\langle p (j) , x\rangle + \langle x, c (p (j))\rangle = 0.$$

Since $p(1), \ldots, p(n)$ form a basis, the last equation implies that for arbitrary x and y

$$\langle y, x\rangle + \langle x, c(y)\rangle = 0. \tag{5.5}$$

6. Affine root system

Let Γ be an affine Dynkin graph. Then the kernel of bilinear symmetric form in $\mathbb{Z}^n$ is one-dimensional and generated by

$$\delta = a_0\alpha_0 + a_1\delta_1 + \cdots + a_n\delta_n.$$

We assume without loss of generality that the vertex α_0 is such that $a_0 = 1$. By removing 0 from Γ we get a Dynkin graph which we denote by Γ^0. In affine case roots can be of two kinds: *real*, if $q(\alpha) = 1$, or *imaginary*, $q(\alpha) = 0$.

Lemma 6.1. *Imaginary roots are all proportional to δ, real roots can be written as $\alpha + m\delta$ for some root δ of Γ^0. Every real root can be obtained from a simple root by the action of the Weyl group W.*

Proof. The first statement is obvious, the second follows from the fact that $q(\alpha) = q(\alpha + m\delta)$, hence the projection on the hyperplane generated by $\alpha_1, \ldots \alpha_n$ maps a root to a root. To prove the last statement, note that r_i maps every positive root different from α_i to a positive root. Let α be a positive real root, $\alpha = a_0\alpha_0 + \cdots + a_n\alpha_n$, and $h(\alpha) = a_0 + a_1 + \cdots + a_n$. Then $(\alpha, \alpha_i) > 0$ at least for one i. But then $h(r_i(\alpha)) < h(\alpha)$. Thus, one can decrease $h(\alpha)$ by application of simple reflection. In the end one can get a root of height 1, which is a simple root. Similarly for negative roots. $\square$

7. Kronecker quiver

In this section we use Coxeter functors to classify indecomposable representation of the quiver $\hat{A}_1 = \bullet \Rightarrow \bullet$. The admissible enumeration of vertices is $1 \Rightarrow 0$, $\delta = \alpha_0 + \alpha_1$. Positive real roots are

$$m\alpha_1 + (m+1)\alpha_0 = -\alpha_1 + (m+1)\delta,\ (m+1)\alpha_1 + m\alpha_0 = \alpha_1 + m\delta,\ m \geq 0.$$

The Coxeter element $c = r_1 r_0$ satisfies

$$c(\alpha_1) = \alpha_1 + 2\delta,\ c(\delta) = \delta.$$

Let $x = m\alpha_1 + l\delta$. If $m > 0$ then $c^{-s}(x)$ is not positive for sufficiently large s. Hence if X is indecomposable of dimension x, then $(\Phi^-)^s X = 0$. If $m < 0$, then $(\Phi^+)^s X = 0$. Thus if $m \neq 0$, then as in the case of Dynkin quiver, X can be obtained from some L_i by application of reflection functor. In particular, we obtain that the dimension of an indecomposable representation is always a root and if this root is real, then the indecomposable with this dimension is unique up to an isomorphism. Indeed, we have either

$$k^m \rightrightarrows^{A}_{B} k^{m+1},$$

where $A = (1_m, 0)\,, B = (0, 1_m)$, or

$$k^{m+1} \rightrightarrows^{C}_{D} k^m,$$

where $C = A^t, D = B^t$.

Classification of indecomposables of dimension $m\delta$ is equivalent to classification of pairs of linear operators $(A, B) : k^m \to k^m$ up to equivalence $(A, B) \sim (PAQ^{-1}, PBQ^{-1})$. Assume that A is invertible, then one may assume that $A = \mathrm{Id}$, and then classify B up to conjugation. Indecomposability of the representation implies that B is equivalent to the Jordan block with some eigenvalue μ. Denote the corresponding representation by ρ_μ. If B is invertible, then A is equivalent to a Jordan block. Denote such representation by σ_μ. One can see that ρ_μ is isomorphic to $\sigma_{\mu^{-1}}$ if $\mu \neq 0$. Now let us prove that at least one of A and B is invertible. Indeed, indecomposability implies that $\operatorname{Ker} A \cap \operatorname{Ker} B = 0$. Hence $A + tB$ is invertible for some t. Without loss of generality one can assume that $A + tB = id$. But then either A or B must be invertible. Thus, we proved that indecomposable representation of dimension $(m, m) = \delta$ are parameterized by a projective line.

For other affine quivers, the situation is more complicated, as there are real roots which remain positive under Coxeter transformation. For example consider the quiver $\widehat{D}_4$

$$\begin{array}{ccccc} & & 5 & & \\ & & \downarrow & & \\ 1 & \to & 2 & \to & 4 \\ & & \uparrow & & \\ & & 3 & & \end{array}$$

Then $c\,(\alpha_1 + \alpha_2 + \alpha_3) = \alpha_4 + \alpha_2 + \alpha_5$, $c^2\,(\alpha_1 + \alpha_2 + \alpha_3) = 0$.

PROBLEM SET # 11
MATH 252

Due November 21.

1. Let Rep (a, b, c) be the space of all representations of the quiver

$$\bullet \to \bullet \leftarrow \bullet$$

with dimension vector (a, b, c). List all orbits in Rep (a, b, c). Show that there is only one open orbit. Describe the open orbit O_X in terms of decomposition of X into direct sum of indecomposable representations.

2. Classify indecomposable representations of the quiver A_n with orientation:

$$\bullet \to \bullet \to \cdots \to \bullet.$$

You can use Gabriel's theorem.

Date: November 14, 2005.

REPRESENTATION THEORY.
WEEK 13.

1. Preprojective and preinjective representations.

The goal of these notes is to obtain a classification of indecomposable representations of affine quivers. It is rather technical and long.

A representation X is called *preprojective* if $(\Phi^+)^s X = 0$ for some s, *preinjective* if $(\Phi^-)^s (X) = 0$ for some s and *regular* if it is not preprojective or preinjective.

Example 1.1. For Kronecker quiver all preinjective indecomposable representations have dimension $\alpha_1 + m\delta$, all preprojective representations have dimension $-\alpha_1 + m\delta$, and regular indecomposable representations have dimension $m\delta$.

Lemma 1.2. *If X is preprojective, then $X = (\Phi^-)^s P$ for some projective P. If X is preinjective, then $X = (\Phi^+)^s I$ for some injective I.*

Proof. Suppose $(\Phi^+)^s X \neq 0$, and $(\Phi^+)^{s+1} X = 0$. Then

$$F_{i-1}^+ \dots F_1^+ \left(\Phi^+\right)^s X = L_i,\ X \cong \left(\Phi^-\right)^s F_1^- \dots F_{i-1}^- (L_i)$$

as in Corollary 4.3 (week 12). Therefore by Corollary 5.4 (week12)

$$X \cong \left(\Phi^-\right)^s (Ae_i).$$

For preinjective similarly. □

Corollary 1.3. *If X is an indecomposable preprojective or preinjective, then* $\dim X$ *is a real root. If X and Y are preprojective indecomposable representations of the same dimension, then $X \cong Y$. An indecomposable preprojective representation is a brick with trivial self-extensions.*

We see from above corollary that preprojective and preinjective indecomposables can be described precisely in terms of reflection functors in the same way as it was done for Dynkin quivers. The next lemma allows "to separate" preinjective, preinjective and regular indecomposable representations.

Lemma 1.4. *If X, Y are indecomposable and X is preprojective, Y is not, then* $\mathrm{Hom}_Q (Y, X) = \mathrm{Ext}^1 (X, Y) = 0$. *If X is preinjective, Y is not, then* $\mathrm{Hom}_Q (X, Y) = \mathrm{Ext}^1 (Y, X) = 0$.

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Proof. Let X be preprojective. Then $X = (\Phi^-)^s P$ for some projective P. Then

$$\operatorname{Ext}^1\left((\Phi^-)^s P, Y\right) = \operatorname{Ext}^1\left(P, (\Phi^+)^s Y\right) = 0$$

by Corollary 5.2 (week12). On the other hand,

$$\left(\Phi^+\right)^{s+1} X = 0,\ Y \cong \left(\Phi^-\right)^{s+1}\left(\Phi^+\right)^{s+1} Y$$

and

$$\operatorname{Hom}_Q\left(X, \left(\Phi^-\right)^{s+1}\left(\Phi^+\right)^{s+1} Y\right) = \operatorname{Hom}_Q\left(\left(\Phi^+\right)^{s+1} X, \left(\Phi^+\right)^{s+1} Y\right) = 0.$$

For preinjective use duality. □

Let now Q be affine and define *defect* X by

$$\operatorname{def}(X) = \langle \delta, x\rangle = -\langle x, \delta\rangle.$$

We write $x \leq y$ if $y - x \in \mathbb{Z}^n_{\geq 0}$

Lemma 1.5. *If $x < \delta$ and X is indecomposable of dimension x, then X is a brick, x is a root and* $\operatorname{Ext}^1(X, X) = 0$.

Proof. If X is not a brick, then there is a brick $Y \subset X$ such that $\operatorname{Ext}^1(Y, Y) \neq 0$ (proven week 11). But then $q(y) \leq 0$, which is impossible as $y < \delta$. Hence X is a brick. Since $q(x) > 0$, we have $\operatorname{Ext}^1(X, X) = 0$ and $q(x) = 1$. □

Lemma 1.6. *There is an indecomposable representation of dimension δ.*

Proof. Pick an orbit O_Z in $\operatorname{Rep}(\delta)$ of maximal dimension. Then Z is indecomposable, because otherwise $Z = X_1 \oplus \cdots \oplus X_p$, where X_i are as in previous lemma, $\operatorname{Ext}^1(X_i, X_j) = 0$ and then $q(z) = p$. □

Lemma 1.7. *If X is regular, then there is s such that $c^s(x) = x$.*

Proof. One can find s such that $c^s(x) = x + l\delta$. But if $l \neq 0$, then $c^{ds}(x) < 0$ for some $d \in \mathbb{Z}$. This contradicts regularity of X. □

Theorem 1.8. *Let X be indecomposable.*

(1) *If X is preprojective, then* $\operatorname{def}(X) < 0$;
(2) *If X is regular, then* $\operatorname{def}(X) = 0$;
(3) *If X is preinjective, then* $\operatorname{def}(X) > 0$.

Proof. Let X be preprojective, Z as in Lemma 1.6. Then $\operatorname{Ext}^1(X, Z) = 0$ by Lemma 1.4. On the other hand, $X = (\Phi^-)^s Ae_i$. Hence

$$\operatorname{Hom}_Q(X, Z) = \operatorname{Hom}_Q\left(\left(\Phi^-\right)^s Ae_i, Z\right) = \operatorname{Hom}_Q\left(Ae_i, \left(\Phi^+\right)^s Z\right) \neq 0,$$

and $\langle x, \delta\rangle > 0$. For preinjective X use duality.

Finally, let X be regular. Assume $\operatorname{def}(X) \neq 0$, say $\operatorname{def}(X) > 0$. Since x is regular $c^s(x) = x$ for some s. Then $y = x + c(x) + \cdots + c^{s-1}(x)$ is c-invariant, therefore $x + c(x) + \cdots + c^{s-1}(x) = m\delta$ by Lemma 3.2 (week 12). But $\langle \delta, c^i(x)\rangle = \langle \delta, x\rangle > 0$ for all $i < s$, hence $\langle \delta, m\delta\rangle > 0$. But $\langle \delta, \delta\rangle = q(\delta) = 0$. Contradiction. □

2. Regular representations

In this section we describe indecomposable regular representations.

We say that a representation is regular if it is a direct sum of indecomposable regular representations.

Theorem 2.1. *If X, Y are regular and $\varphi \in \mathrm{Hom}_Q(X, Y)$ then $\mathrm{Im}\,\varphi$, $\mathrm{Ker}\,\varphi$, $\mathrm{Coker}\,\varphi$ are regular. If*

$$0 \to X \to Z \to Y \to 0$$

is an exact sequence then Z is regular.

Proof. By Lemma 1.4 $\mathrm{Im}\,\varphi$ does not have preinjective summand and preprojective summand. By the same reason $\mathrm{Ker}\,\varphi$ does not have preinjective summand and def $(\mathrm{Ker}\,\varphi) = \mathrm{def}\,(X)$-def $(\mathrm{Im}\,\varphi) = 0$. Hence $\mathrm{Ker}\,\varphi$ is regular. Similarly, $\mathrm{Coker}\,\varphi$ is regular.

Finally, suppose Z has a preprojective direct summand Z_i. This is impossible by the long exact sequence

$$\mathrm{Hom}_Q(X, Z_i) = 0 \to \mathrm{Hom}_Q(Z, Z_i) \to \mathrm{Hom}_Q(Y, Z_i) = 0.$$

Similarly, Z could not have preinjective direct summand. □

A regular representation X is called *regular simple* if X has no proper non-trivial regular subrepresentations. By Theorem 2.1 a regular simple representation is a brick, hence $q(x) \leq 1$ and x is a root.

Example 2.2. For Kronecker quiver a representation $k \rightrightarrows^{\mu}_{\lambda} k$ is simple for any $(\lambda, \mu) \neq (0, 0)$. One can see easily from classification of indecomposables that those are all regular simple representations.

Example 2.3. For the quiver $\widehat{D}_4$ an indecomposable representation

$$\begin{array}{ccccc} & & k & & \\ & & \downarrow^{\tau_2} & & \\ k & \xrightarrow{\tau_1} & k^2 & \xrightarrow{\tau_4} & k \\ & & \uparrow^{\tau_3} & & \\ & & k & & \end{array}$$

is regular simple iff $\mathrm{Im}\,\tau_i \neq \mathrm{Im}\,\tau_j$ for all $i \neq j$.

Let $\delta = a_0\alpha_0 + \cdots + a_n\alpha_n$. Without loss of generality we may assume that $a_0 = 1$. Let $P = Ae_0$, $p = \dim P$ and R be the indecomposable preprojective representation of dimension $r = p + \delta$. There are the following identities

$$\langle p, \delta\rangle = \langle r, \delta\rangle = 1, \ \langle p, r\rangle = 2, \ \langle r, p\rangle = 0.$$

Since $\mathrm{Ext}^1(P, R) = 0$, $\mathrm{Hom}_Q(P, R) = k^2$.

Lemma 2.4. *Let $\theta \in \mathrm{Hom}_Q(P, R)$. If $\theta \neq 0$, then θ is injective. Let $\eta \in \mathrm{Hom}_Q(R, P)$. If $\eta \neq 0$ then η is surjective.*

Proof. Both $\operatorname{Ker}\theta$ and $\operatorname{Im}\theta$ are preprojective, because P and R are preprojective. Since $-1 = \operatorname{def}(P) = \operatorname{def}(\operatorname{Ker}\theta) + \operatorname{def}(\operatorname{Im}\theta)$, either $\operatorname{Ker}\theta$ or $\operatorname{Im}\theta$ is zero. The second statement is similar. □

Corollary 2.5. $\operatorname{Hom}_Q(R,P) = \operatorname{Ext}^1(R,P) = 0$.

Proof. Let $\eta \in \operatorname{Hom}_Q(R,P)$ be surjective, then $\eta = 0$ since P is projective and must split as a direct summand of R, but R is indecomposable. Hence $\operatorname{Hom}_Q(R,P) = 0$. Since $\langle r,p\rangle = 0$, $\operatorname{Ext}^1(R,P) = 0$. □

Corollary 2.6. *Let $\theta \in \operatorname{Hom}_Q(P,R)$, $\theta \neq 0$. Then $Z_\theta = \operatorname{Coker}\theta$ is indecomposable regular.*

Proof. Use the sequence

$$0 \to P \to R \to Z_\theta \to 0.$$

The long exact sequence

$$0 = \operatorname{Hom}_Q(R,P) \to \operatorname{Hom}_Q(R,R) \to \operatorname{Hom}_Q(R,Z_\theta) \to \operatorname{Ext}^1(R,P) = 0$$

implies $\operatorname{Hom}_Q(R,Z_\theta) = \operatorname{End}_Q(R) = k$. The long exact sequence

$$0 \to \operatorname{Hom}_Q(Z_\theta,Z_\theta) \to \operatorname{Hom}_Q(R,Z_\theta) = k \to \dots$$

implies $\operatorname{Hom}_Q(Z_\theta,Z_\theta) = k$. Hence Z_θ is indecomposable. Since $\operatorname{def}(Z_\theta) = 0$, Z_θ is regular. □

Lemma 2.7. *Let X be regular indecomposable and $x_0 \neq 0$, where $x_0 = \dim X_0$. Then there exists $\theta \in \operatorname{Hom}_Q(P,R)$ such that $\operatorname{Hom}_Q(Z_\theta, X) \neq 0$.*

Proof. First note that

$$\dim \operatorname{Hom}_Q(P,X) = x_0 = \dim(Q,X)$$

since $\langle p,x\rangle = \langle q,x\rangle = x_0$.

Any $\theta \in \operatorname{Hom}_Q(P,R)$ defines the linear map

$$\theta^* \colon \operatorname{Hom}_Q(R,X) \to \operatorname{Hom}_Q(P,X).$$

Since $\dim \operatorname{Hom}_Q(P,R) = 2$, one can find $\theta \in \operatorname{Hom}_Q(P,R)$ such that θ^* is not invertible. Then there is $\varphi \in \operatorname{Hom}_Q(R,X)$ such that $\theta^*(\varphi) = \varphi \circ \theta = 0$. Then $\varphi(\theta(P)) = 0$, and φ is well defined homomorphism $Z_\theta \to X$. □

Corollary 2.8. *Let X be regular simple, then $x \leq \delta$.*

Proof. We already know that x is a root. If $x_0 \neq 0$, then $\operatorname{Hom}_Q(Z_\theta, X) \neq 0$ for some θ and therefore X is a quotient of Z_θ, hence $x \leq \delta$. If $x_0 = 0$, then $x < \delta$. □

Example 2.9. In case of $\widehat{D}_4$ the regular simples have dimensions $\alpha_1+\alpha_2+\alpha_3$, $\alpha_1+\alpha_2+\alpha_4$, $\alpha_1+\alpha_2+\alpha_3$, $\alpha_3+\alpha_2+\alpha_4$, $\alpha_3+\alpha_2+\alpha_5$, $\alpha_4+\alpha_2+\alpha_5$ or δ. There is exactly one simple for each real root and one-parameter family for δ.

Our next step is to describe extensions between regular simple representations.

Lemma 2.10. *Let X and Y be two regular simples, then* $\mathrm{Hom}_Q(X,Y)=k$ *iff* $X\cong Y$ *and* $\mathrm{Ext}^1(X,Y)=k$ *iff* $Y\cong\Phi^+X$. *Otherwise* $\mathrm{Ext}^1(X,Y)=\mathrm{Hom}_Q(X,Y)=0$.

Proof. The statement about Hom is trivial since any nonzero $\varphi\in\mathrm{Hom}_Q(X,Y)$ is an isomorphism . To prove the statement about Ext^1, use (5.5) from lecture notes week 12. First, assume that $Y\ncong\Phi^+X,X$, then

$$\langle x,y\rangle=\dim\mathrm{Hom}_Q(X,Y)-\dim\mathrm{Ext}^1(X,Y)\le 0,$$
$$\langle y,c(x)\rangle=\dim\mathrm{Hom}_Q\left(Y,\Phi^+X\right)-\dim\mathrm{Ext}^1\left(Y,\Phi^+X\right)\le 0.$$

Since $\langle x,y\rangle+\langle y,c(x)\rangle=0$, $\mathrm{Ext}^1(X,Y)=0$.

Now assume that $X\cong Y$. If x is a real root, then $\mathrm{Ext}^1(X,X)=0$, and $\langle x,x\rangle=1$. Then $\langle x,c(x)\rangle=-1$, which implies $\mathrm{Ext}^1(X,\Phi^+X)=k$. If $x=\delta$, then

$$\mathrm{Hom}_Q(X,X)=\mathrm{Ext}^1(X,X)=k.$$

□

Corollary 2.11. *If X is regular simple and $x<\delta$, then $(\Phi^+)^s X\cong X$ for some s. If $x=\delta$, then $\Phi^+X\cong X$.*

The minimal number s such that $(\Phi^+)^s X\cong X$ is called the *period* of X. Regular simples can be divided in orbits under action of Φ^+.

In the category of regular representations one can define Jordan-Hölder series and regular length, and regular series is again unique up to permutation.

The following theorem gives a complete description of indecomposable regular representations.

Theorem 2.12. *Let X be regular indecomposable then there is a unique filtration*

$$0\subset X_1\subset X_2\subset\cdots\subset X_r=X \tag{2.1}$$

such that $Y_i\cong X_i/X_{i-1}$ is regular simple and

$$Y_{i-1}\cong\Phi^+(Y_i),\ \mathrm{Ext}^1(Y_r,X_{r-1})\cong\mathrm{Ext}^1(Y_r,Y_{r-1})\cong k.$$

Moreover, $\mathrm{Ext}^1(Z,X_{r-1})=0$ *for any regular simple Z not isomorphic to Y_r.*

Proof. We prove Theorem by induction on the regular length of X. Check yourself case $r=2$. If X has length r then it has a filtration (2.1), although it might be not unique. Assume first that X_{r-1} is indecomposable. Then it satisfies all the statements of Theorem by induction assumption. Consider the exact sequence

$$0\to X_{r-2}\to X_{r-1}\to Y_{r-1}\to 0$$

and induced long exact sequence

$$\mathrm{Hom}_Q(Y_r,X_{r-1})\xrightarrow{a}\mathrm{Hom}_Q(Y_r,Y_{r-1})\xrightarrow{b}\mathrm{Ext}^1(Y_r,X_{r-2})\xrightarrow{c}\mathrm{Ext}^1(Y_r,X_{r-1})\xrightarrow{d}\mathrm{Ext}^1(Y_r,Y_{r-1})\to 0.$$

We claim that d is an isomorphism. If $\mathrm{Ext}^1(Y_r,X_{r-2})=0$, then it is trivial. Assume that $\mathrm{Ext}^1(Y_r,X_{r-2})\neq 0$. By induction assumption $Y_r\cong\Phi^+Y_{r-2}\cong Y_{r-1}$. Then $a=0$

by uniqueness of filtration for X_{r-1}, b must be an isomorphism, c forced to be zero, and therefore d is an isomorphism.

Now we prove that X_{r-1} is indecomposable. Assume the opposite. Then $X_{r-1} = Z_1 \oplus \cdots \oplus Z_s$ where each Z_i is indecomposable and satisfies the statement of Theorem. Assume that Z_1 has maximal length among Z_i. Then there is a surjective homomorphism $p_i \colon Z_1 \to Z_i$ for each i, and this homomorphism induces the isomorphism

$$p_{i*} \colon \operatorname{Ext}^1(Y_r, Z_1) \to \operatorname{Ext}^1(Y_r, Z_i) \cong k.$$

Consider the exact sequence

$$0 \to \bigoplus Z_i \to X \to Y_r \to 0. \tag{2.2}$$

It is induced by some $\psi \in \operatorname{Ext}^1(Y_r, \bigoplus Z_i)$. But $\psi = \psi_1 + \cdots + \psi_s$, $\psi_i \in \operatorname{Ext}^1(Y_r, Z_i)$. Hence each $\psi_i = c_i p_{i*}(\psi_1)$. Let

$$Z' = \left\{ (z_1, \ldots, z_s) \in \bigoplus Z_i \mid z_i = c_i p_i(z_1) \right\},$$

then one can find $X' \subset X$ such that

$$0 \to Z' \to X' \to Y_r \to 0,$$

is a subsequence of (2.2). Then X' splits as a summand in X. Contradiction.

Check now that X has a unique regular maximal submodule X_{r-1} (that implies the uniqueness of filtration). Consider the exact sequence

$$0 \to X_{r-1} \to X \xrightarrow{f} Y_r \to 0.$$

Let X' be another maximal submodule, then $f(X') = Y_r$ and we have an an exact sequence

$$0 \to X_{r-1} \cap X' \to X' \to Y_r \to 0.$$

However, the regular length of X' is $r-1$, hence $X_{r-1} \cap X' = X_{r-2}$. Therefore $X/X_{r-2} \cong Y_r \oplus Y_{r-1}$. But the sequence

$$0 \to Y_{r-1} \to X/X_{r-2} \to Y_r \to 0$$

does not split since it is induced by a non-zero element in $\operatorname{Ext}^1(Y_r, X_{r-1}) \cong \operatorname{Ext}^1(Y_r, Y_{r-1})$. Contradiction. □

Corollary 2.13. *Let $Y_1, \ldots, Y_r$ be a sequnce of simple regular representations such that $Y_{i-1} \cong \Phi^+ Y_i$. Then there exists a unique up to an isomorphism regular indecomposable X with filtration $0 \subset X_1 \subset X_2 \subset \cdots \subset X_r = X$ such that $X_i/X_{i-1} \cong Y_i$.*

Proof. Construct X inductively using the isomorphism

$$\operatorname{Ext}^1(Y_{t+1}, X_t) \cong \operatorname{Ext}^1(Y_{t+1}, Y_t) \cong k.$$

□

As follows from Theorem 2.12 simple regular subquotients of an indecomposable regular representation belong to one orbit of Φ^+. Thus, each orbit of Φ^+ in the set of simple regular representations defines a family of indecomposables called a *tube*.

Lemma 2.14. *Let X be regular indecomposable, then* $\dim X$ *is a root.*

Proof. Lemma follows from Theorem 2.12 and the following fact. Let α, β be real roots. Then $(\alpha, \beta) = -1$ implies $\alpha + \beta$ is a real root, $(\alpha, \beta) = -2$ implies $\alpha + \beta$ is an imaginary root. □

Lemma 2.15. *Every tube contains exactly one indecomposable representation isomorphic to Z_θ.*

Proof. Let X be simple regular of period s, i.e. $(\Phi^+)^s X \cong X$. If $x = \dim X$, then

$$x + c(x) + \cdots + c^{s-1}(x) = m\delta. \tag{2.3}$$

Choose $y = c^i(x)$ such that $y_0 \neq 0$. Let $Y = (\Phi^+)^i X$. By Lemma 2.7 there exist $\theta \in \mathrm{Hom}_Q(P, R)$ and a non-zero homomorphism $\varphi : Z_\theta \to Y$. Then the indecomposable Z_θ has the filtration

$$0 \subset Z_1 \subset Z_2 \subset \cdots \subset Z_s = Z_\theta$$

such that $Z_s/Z_{s-1} \cong Y$. Then Z_θ is in a tube. Moreover, one can see now that $m = 1$ in (2.3) and therefore Z_θ is unique. □

3. Indecomposable representations of affine quivers

In the next theorem we summarize our results about affine quivers.

Theorem 3.1. *Let Q be an affine quiver, then dimension of every indecomposable representation of Q is a root. If α is a real root, then there exists exactly one (up to an isomorphism) indecomposable representation of dimension α. If $\alpha = m\delta$, then there are infinitely many indecomposable representations of dimension α.*

Proof. Let α be the dimension of an indecomposable representation X. If $\langle \alpha, \delta \rangle \neq 0$, then X is preprojective or preinjective, and α is a real root by Corollary 1.3. If $\langle \alpha, \delta \rangle = 0$, then X is regular and α is a root by Lemma 2.14. The uniqueness of X follows from Theorem 2.12. We also have to prove that for each α there is an indecomposable of dimension α. If $\langle \alpha, \delta \rangle > 0$, choose the minimal i and s such that $r_i \dots r_1 c^s(\alpha) < 0$, then put $X = (\Phi^-)^s \circ F_1^- \circ \cdots \circ F_{i-1}^-(L_i)$. The case $\langle \alpha, \delta \rangle < 0$ is similar. Let $\langle \alpha, \delta \rangle = 0$. Assume first that $\alpha < \delta$. Construct X as an orbit of maximal dimension in $\mathrm{Rep}(\alpha)$. If $\alpha = \beta + m\delta$, for some $\beta < \delta$, construct an indecomposable Y of dimension β, and extend it using the description of a tube. □

Example 3.2. Consider the quiver $\hat{\mathrm{A}}_n$. The indecomposable representations of real dimension and regular indecomposables of imaginary dimension with period greater than 1 are enumerated by counterclockwise walks around the quiver (ignoring the orientation). A basis $\{v_i\}$ in representation X is enumerated by vertices which appear in a walk. For each γ put $\rho_\gamma(v_i) = v_{i+1}$ if the orientation of γ is counterclockwise and $\rho_\gamma(v_{i+1}) = v_i$ if the orientation of γ is clockwise. The last vector in the walk goes to 0 if γ is counterclockwise oriented.

If X has imaginary dimension and X is in a tube of period 1, then X can be described by the following construction. Put $X_i \cong k^m$ for all i, $\rho_\gamma = \mathrm{Id}$ for all γ except one arrow σ (it does not matter which one you choose). Let ρ_σ be a Jordan block with non-zero eigenvalue.

Let Q and Q' two different quivers in the graph $\hat{\mathrm{A}}_n$. It is not always possible to obtain Q' from Q using reflection functors. If Q and Q' have the same number of clockwise (hence counterclockwise) arrows, one can obtain Q' from Q by a chain of reflections.

Check yourself that if Q has p counterclockwise arrows, then Q has one tube of period $p-1$ and one tube of period $n-p-1$.

REPRESENTATION THEORY.
WEEK 14

1. Applications of quivers

Two rings A and B are *Morita equivalent* if the categories of $A-$ modules and B-modules are equivalent. A projective finitely generated A-module P is a *projective generator* if any other projective finitely generated A-module is isomorphic to a direct summand of $P^{\oplus n}$ for some n.

Theorem 1.1. *A and B are Morita equivalent iff there exists a projective generator P in $A-\ \mathrm{mod}$ such that $B \cong \mathrm{End}_A(P)$. The functor $X \mapsto \mathrm{Hom}_A(P, X)$ establishes the equivalence between $A-\ \mathrm{mod}$ and $B-\ \mathrm{mod}$.*

For the proof see, for example, Bass "Algebraic K-theory".

Assume now that C is a finite-dimensional algebra over algebraically closed field k. Let $P_1, \dots, P_n$ be a set of representatives of isomorphism classes of indecomposable projective C-modules. Then $P = P_1 \oplus \cdots \oplus P_n$ is a projective generator, and $A = \mathrm{End}_C(P)$ is Morita equivalent to C.

Example 1.2. Let C be semisimple, then $C \cong \mathrm{Mat}_{m_1}(k) \times \cdots \times \mathrm{Mat}_{m_n}(k)$, and $A \cong k^n$. Let

$$C = \left\{ \left(\begin{smallmatrix} X & Y \\ 0 & Z \end{smallmatrix}\right) \in \mathrm{Mat}_{p+q}(k) \mid X \in \mathrm{Mat}_p(k), Y \in \mathrm{Mat}_{p,q}(k), Z \in \mathrm{Mat}_q(k) \right\}.$$

Then

$$A = \left\{ \left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right) \mid x, y, z \in k \right\}.$$

Let R be the radical of C. Then each indecomposable projective P_i has the filtration $P_i \supset RP_i \supset R^2P_i \supset \cdots \supset 0$ such that $R^jP_i/R^{j+1}P_i$ is semisimple for all j. Recall that P_i/RP_i is simple (lecture notes 9), hence $\mathrm{Hom}_C(P_i, P_j/RP_j) = 0$ if $i \neq j$. Define the quiver Q in the following way. Vertices are enumerated by indecomposable projective modules $P_1, \dots, P_n$, the number of arrows $i \to j$ equals $\dim \mathrm{Hom}_C(P_i, RP_j/R^2P_j)$. We construct a surjective homomorphism $\phi \colon k(Q) \to A$. (This construction is not canonical). Fist set $\phi(e_i) = \mathrm{Id}_{P_i}$. Let $\gamma_1, \dots, \gamma_s$ be the set of arrows from i to j, choose a basis $\eta_1, \dots, \eta_s \in \mathrm{Hom}_C(P_i, RP_j/R^2P_j)$, each η_l can be lifted to $\xi_l \in \mathrm{Hom}_C(P_i, RP_j)$ as P_i is projective. Define $\phi(\gamma_l) = \xi_l$. Now ϕ extends in the unique way to the whole $k(Q)$ since $k(Q)$ is generated by idempotents e_i and arrows.

Since ϕ is surjective, then $A \cong k(Q)/I$ for some two-sided ideal $I \subset k(Q)$. The pair Q and an ideal I in $k(Q)$ is called a *quiver with relations.* The problem of classification of indecomposable C-modules is equivalent to the problem of classification

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of indecomposable representations of Q satisfying relations I. In some cases such quiver approach is very useful.

Example 1.3. Let k be the algebraic closure of $\mathbb{F}_3$ and $C = k[S_3]$. In lecture notes 9 we showed that C has two indecomposable projectives $P_+ = \mathrm{Ind}_{S_2}^{S_3}$ triv and $P_- = \mathrm{Ind}_{S_2}^{S_3}$ sgn. The quiver Q is

$$\bullet \iff {}^{\alpha}_{\beta}\bullet$$

with relations $\alpha\beta\alpha = 0$, $\beta\alpha\beta = 0$. The quiver itself is $\hat{\mathrm{A}}_2$, indecomposable representations have dimensions (m,m), $(m+1,m)$ and $(m,m+1)$. Since we have the precise description, it is not difficult to see that only six indecomposable representations satisfy the relations. They are

$$k \iff 0;\ 0 \iff k;\ k \iff k,\ \alpha = 1, \beta = 0 \text{ or } \alpha = 0, \beta = 1,$$
$$k^2 \iff k,\ \alpha = (10)\,, \beta = \binom{0}{1};\ k \iff k^2,\ \alpha = \binom{0}{1}\,,\ \beta = (10)\,.$$

The first two representations correspond to irreducible representations triv and sgn, the last two are projectives. Two representations of dimension (1,1) correspond to the quotients of P_+ and P_- by the minimal submodules.

In fact one can apply the quiver approach to any category $\mathcal{C}$ which satisfies the following conditions

(1) All objects have finite length;
(2) Any object has a projective resolution;
(3) For any two objects X, Y, $\mathrm{Hom}\,(X, Y)$ is a vector space over an algebraically closed field k.

We do not need the assumption that the number of simple or projective objects is finite. We illustrate this in the following example.

Example 1.4. Let Λ be the Grassmann algebra with two generators, i.e. $\Lambda = k < x, y > / (x^2, y^2, xy + yx)$. Consider the $\mathbb{Z}$-grading $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \Lambda_2$, where $\Lambda_0 = k$, Λ_1 is the span of x and y, $\Lambda_2 = kxy$. Let $\mathcal{C}$ denote the category of graded Λ-modules. In other words, objects are Λ-modules $M = \bigoplus_{i\in\mathbb{Z}} M_i$, such that $\Lambda_i M_j \subset M_{i+j}$ and morphisms preserve the grading. All projective modules are free. An indecomposable projective module P_i is isomorphic to Λ with shifted grading $\deg(1) = i$. Thus, the quiver Q has infinitely many vertices enumerated by $\mathbb{Z}$:

$$\cdots \Leftarrow^{\alpha_i}_{\beta_i} \bullet \Leftarrow^{\alpha_{i+1}}_{\beta_{i+1}} \bullet \Leftarrow^{\alpha_{i+2}}_{\beta_{i+2}} \bullet \Leftarrow^{\alpha_{i+3}}_{\beta_{i+3}} \cdots$$

Here α_{i+1}, $\beta_{i+1} \in \mathrm{Hom}\,(P_{i+1}, P_i)$, $\alpha_{i+1}(1) = x$, $\beta_{i+1}(1) = y$. Relations are $\alpha_i\alpha_{i+1} = \beta_i\beta_{i+1} = 0$, $\alpha_i\beta_{i+1} + \beta_i\alpha_{i+1} = 0$.

Let us classify the indecomposable representations of above quiver. Assume first that, that there exists $v \in X_{i+1}$ such that $\alpha_i\beta_{i+1} \neq 0$, Then the subrepresentation V spanned by v, $\alpha_{i+1}v$, $\beta_{i+1}v$, $\alpha_i\beta_{i+1}v$ splits as a direct summand in X. If X is indecomposable, then $X = V$. The corresponding object in $\mathcal{C}$ is P_{i+1}.

Now assume that $\alpha_i \beta_{i+1} X_{i+1} = 0$ for any $i \in \mathbb{Z}$. That is equivalent to putting the new relations for Q: every path of length 2 is zero. Consider the subspaces

$$W_i = \operatorname{Im} \alpha_{i+1} + \operatorname{Im} \beta_{i+1} \subset X_i,\ Z_{i+1} = \operatorname{Ker} \alpha_{i+1} \cap \operatorname{Ker} \beta_{i+1} \subset X_{i+1}.$$

One can find $U_i \subset X_i$ and $Y_{i+1} \subset X_{i+1}$ such that $X_i = U_i \oplus W_i$, $X_{i+1} = Z_{i+1} \oplus Y_{i+1}$. Check that $W_i \oplus Y_{i+1}$ is a subrepresentation, which splits as a direct summand in X. If X is indecomposable and $W_i \neq 0$, then $X = W_i \oplus Y_{i+1}$. Thus, we reduced our problem to Kronecker quiver $\bullet \Leftarrow \bullet$! There is the obvious bijection between indecomposable non-projective objects from $\mathcal{C}$ and the pairs (Y, i), where Y is an indecomposable representation of Kronecker quiver, $i \in \mathbb{Z}$ (defines the grading).

Remark 1.5. The last example is related to the algebraic geometry as the derived category of $\mathcal{C}$ is equivalent to the derived category of coherent sheaves on $\mathbb{P}^1$.

Remark 1.6. If in the last example we increase the number of generators in Λ, then the problem becomes wild (definition below).

Let C be a finite-dimensional algebra. We say that C is *finitely represented* if C has finitely many indecomposable representations. We call C *tame* if for each $d \subset \mathbb{Z}_{>0}$, there exist a finite set $M_1, \ldots, M_r$ of $C - k[x]$ bimodules (free of rank d over $k[x]$) such that every indecomposable representation of C of dimension d is isomorphic to $M_i \otimes_{k[x]} k[x] / (x - \lambda)$ for some $i \leq r$, $\lambda \in k$. Finally, C is *wild* if there exists a $C - k < x, y >$ bimodule M such that the functor $X \mapsto M \otimes_{k<x,y>} X$ preserves indecomposability and is faithful. We formulate here without proof the following results.

Theorem 1.7. *Every finite-dimensional algebra over algebraically closed field k is either finitely represented or tame or wild*

Theorem 1.8. *Let Q be a connected quiver without oriented cycles. Then $k(Q)$ is finitely represented iff Q is Dynkin, $k(Q)$ is tame iff Q is affine.*

Theorem 1.9. *Let Alg_n be the algebraic variety of all n-dimensional algebras over k. Then the set of finitely represented algebras is Zariski open in Alg_n.*

2. Frobenius algebras

Let A be a finite-dimensional algebra over k. Recall that we denote by D the functor $\mod - A \to A - \mod$, such that $D(X) = X^*$. Recall also that D maps projective modules to injective and vice versa.

A finite-dimensional A algebra over k is called a *Frobenius algebra* if $D(A_A)$ is isomorphic to A, where A_A is the right A-module over itself.

Theorem 2.1. *The following conditions on A are equivalent*

1. *A is a Frobenius algebra;*
2. *There exists a non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ on A such that $\langle ab, c \rangle = \langle a, bc \rangle$;*

(3) *There exists* $\lambda \in A^*$ *such that* $\operatorname{Ker}\lambda$ *does not contain non-trivial left or right ideals.*

Proof. A form $\langle\cdot,\cdot\rangle$ gives an isomorphism $\mu\colon A \to A^*$ by the formula $x \to \langle\cdot, x\rangle$. The condition $\langle ab, c\rangle = \langle a, bc\rangle$ is equivalent to μ being a homomorphism of modules. A linear functional λ can be constructed by $\lambda(x) = \langle 1, x\rangle$. Conversely, given λ, one can define $\langle x, y\rangle = \lambda(xy)$. The condition $\operatorname{Ker}\lambda$ does not contains non-trivial one-sided ideals is equivalent to the condition that the left and right kernels of $\langle\cdot,\cdot\rangle$ are zero. □

Lemma 2.2. *Let A be a Frobenius algebra. An A-module X is projective iff it is injective.*

Proof. A projective module X is a direct summand of a free module, but a free module is injective as $D(A_A)$ is isomorphic to A. Hence, X is injective. By duality an injective module is projective. □

Example 2.3. A group algebra $k(G)$ is Frobenius. Take

$$\lambda\left(\sum_{g\in G} a_g g\right) = a_1.$$

The corresponding bilinear form is symmetric.

A Grassmann algebra $\Lambda = k < x_1, \dots, x_n > / (x_i x_j + x_j x_i)$ is Frobenius. Put

$$\lambda\left(\sum_{i_1<\dots<i_k} c_{i_1\dots i_k} x_{i_1}\dots x_{i_k}\right) = c_{12\dots n}.$$

In a sense Frobenius algebras generalize group algebras. For example, if $T \in \operatorname{Hom}_k(X, Y)$ for two $k(G)$-modules X and Y then

$$\bar{T} = \sum_{g\in G} gTg^{-1} \in \operatorname{Hom}_G(X, Y).$$

This idea of taking average over the group is very important in representation theory. It has an analog for Frobenius algebras.

Choose a basis $e_1, \dots, e_n$ in a Frobenius algebra A. Let $f_1, \dots, f_n$ be the dual basis, i.e.

$$\langle f_i, e_j\rangle = \delta_{ij}. \tag{2.1}$$

Every $a \in A$ can be written

$$a = \sum \langle f_i, a\rangle e_i = \sum \langle a, e_i\rangle f_i. \tag{2.2}$$

and

$$\sum ae_i \otimes f_i = \sum \langle f_j, ae_i\rangle e_j \otimes f_i = \sum \langle f_j a, e_i\rangle e_j \otimes f_i = \sum e_j \otimes f_j a. \tag{2.3}$$

Lemma 2.4. *Let X and Y be A-modules, $T \in \operatorname{Hom}_k(X, Y)$. Then $\bar{T} = \sum e_i T f_i \in \operatorname{Hom}_A(X, Y)$.*

Proof. Direct calculation using (2.2) and (2.3).

□

Example 2.5. If $A = k(G)$, the dual bases can be chosen as $\{g\}_{g\in G}$ and $\{g^{-1}\}_{g\in G}$. Hence $\bar{T} = \sum gTg^{-1}$.

In Frobenius algebra one can use the following criterion of projectivity.

Theorem 2.6. *An A-module X is injective (hence projective) if there exists $T \in \mathrm{End}_k(X)$ such that $\bar{T} = \mathrm{Id}$.*

Proof. First, assume the existence of T. We have to show that X is injective, in other words, for any embedding $\varepsilon\colon X \to Y$ there exists $\pi \in \mathrm{Hom}_A(Y, X)$ such that $\pi \circ \varepsilon = Id$. There exists $p \in \mathrm{Hom}_k(Y, X)$ such that $p \circ \varepsilon = Id$. Put $\pi = \sum e_i T p f_i$. Then for any $x \in X$ we have

$$\pi(\varepsilon(x)) = \sum e_i T p f_i (\varepsilon(x)) = \sum e_i T(p\varepsilon(f_i x)) = \sum e_i T(f_i x) = \bar{T}x = \mathrm{Id}\,.$$

Here we use $f_i\varepsilon = \varepsilon f_i$. By Lemma 2.4 $\pi \in \mathrm{Hom}_A(X, Y)$.

Now assume that X is injective. Define the map $\delta\colon X \to A \otimes_k X$ by the formula

$$f(x) = \sum e_i \otimes f_i x.$$

Then $f \in \mathrm{Hom}_A(X, A \otimes_k X)$ by (2.3). It is obvious that f is injective. Thus, we may consider X as a submodule of X, moreover X is a direct summand because X is injective. So we have a projector $\tau : A \otimes_k X \to X$. Let $S \in \mathrm{Hom}_k(A \otimes_k X, A \otimes_k X)$ be defined by the formula

$$S(a \otimes x) = \langle 1, a\rangle\, 1 \otimes x.$$

Then

$$\bar{S}(a \otimes x) = \sum e_i S(f_i a \otimes x) = \sum \langle 1, f_i a\rangle\, e_i \otimes x = \sum \langle f_i, a\rangle\, e_i \otimes x = a \otimes x$$

due to (2.2). Put $T = \tau \circ S \circ \delta$. Then $\bar{T} = \mathrm{Id}$. □

3. Relative projective and injective modules in group algebra

Let H be a subgroup of a group G. A $k(G)$-module X is *H-injective* if any exact sequence of $k(G)$-modules

$$0 \to X \to Y \to Z \to 0,$$

which splits over $k(H)$, splits over $k(G)$.

In the similar way one defines *H-projective* module.

Let $\{g_1, \dots, g_r\}$ be a set of representatives in the set of left cosets G/H. For any $k(G)$-modules X, Y, and $T \in \mathrm{Hom}_H(X, Y)$ put

$$\bar{T} = \sum g_i T g_i^{-1}.$$

Prove yourself the following

Lemma 3.1. *$\bar{T}$ does not depend on a choice of representatives and $\bar{T} \in \mathrm{Hom}_G(X, Y)$.*

Theorem 3.2. *The following conditions on $k(G)$-module X are equivalent*

1. *X is H-injective;*
2. *X is a direct summand in $\mathrm{Ind}_H^G X$;*
3. *X is H-projective;*
4. *There exists $T \in \mathrm{End}_H(X)$ such that $\bar{T} = \mathrm{Id}$.*

Proof. This theorem is very similar to Theorem 2.6. To prove $1 \Rightarrow 2$ check that $\delta\colon X \to \mathrm{Ind}_H^G X$ defined by the formula

$$\delta(x) = \sum g_i \otimes g_i^{-1} x,$$

defines an embedding of X. By injectivity X is a direct summand of $\mathrm{Ind}_H^G X$.

To prove $3 \Rightarrow 2$ use the projection $\mathrm{Ind}_H^G X \to X$ defined by $g \otimes x \mapsto gx$.

Now prove $2 \Rightarrow 4$. Define $S\colon \mathrm{Ind}_H^G X \to \mathrm{Ind}_H^G X$ by

$$S\left(\sum g_i \otimes x_i\right) = 1 \otimes x_1,$$

here we assume that $g_1 = 1$. Check that $S \in \mathrm{End}_H\left(\mathrm{Ind}_H^G X\right)$ and $\bar{S} = Id$. Then obtain $T = \tau \circ S \circ \delta$, where $\tau\colon \mathrm{Ind}_H^G X \to X$ be the projection such that $\tau \circ \delta = \mathrm{Id}$.

Prove yourself $4 \Rightarrow 1$ and $4 \Rightarrow 3$ similarly to the first part of the proof of Theorem 2.6.

□

The following corollary is important for us. Let p be prime. Recall that if $|G| = p^s r$ with $(p, r) = 1$, then there exists a subgroup P of order p^s. It is called a Sylow subgroup. Two Sylow p-subgroups are conjugate in G.

Corollary 3.3. *Let $\mathrm{char}\, k = p$ and P be a Sylow p-subgroup. Then every $k(G)$-module X is P-injective.*

Proof. We have to check condition (4) from Theorem 3.2. But $r = [G : P]$ is invertible in k. So we can put $T = \frac{1}{r}\mathrm{Id}$. □

4. Finitely represented group algebras

Let $\mathrm{char}\, k = p$, $|G| = p^s r$ with $(p, r) = 1$.

Lemma 4.1. *Let H be a cyclic p-group, i.e. $|H| = p^s$. Then there are exactly p^s isomorphism classes of indecomposable representations of H over k, exactly one for each dimension. More precisely each indecomposable L_m of dimension $m \leq p^s$ is isomorphic to $k(H)/(g-1)^m$, where g is a generator of H.*

Proof. Since $k(H) \cong k[\alpha]/\alpha^{p^s}$, where $\alpha = g - 1$, the corresponding quiver is the loop quiver with one relation $\alpha^{p^s} = 0$. Hence α is a nilpotent Jordan block of order $\leq p^s$. □

Theorem 4.2. *If a Sylow p-subgroup of G is cyclic, then $k(G)$ is finitely represented. Moreover, the number of indecomposable $k(G)$-modules is not greater than $|G|$.*

Proof. By Corollary 3.3 every indecomposable $k(G)$-module is P-injective. Therefore, any indecomposable X is a direct summand in $\operatorname{Ind}_P^G L_i$ for some i. Clearly, the number of such direct summands is finite. Now we will obtain the upper bound on the number of indecomposable representations. Let X be an indecomposbale $k(G)$-module, then by injectivity of X, X is a direct summand in $\operatorname{Ind}_P^G X$. Decompose X into a direct sum of indecomposable $k(P)$-modules, then X must be a direct summand in $\operatorname{Ind}_P^G L_i$ for some P-indecomposable summand L_i of X. Hence $\dim X \geq \dim L_i = i$. So if $\dim X = i$, then X can be realized as a summand in $\operatorname{Ind}_P^G (L_j)$ for some $j \leq i$. To calculate the total number of non-isomorphic indecomposable $k(G)$-modules, we can count in each $\operatorname{Ind}_P^G L_i$ only indecomposable $k(G)$-components of dimension $\geq i$ since others are realized in $\operatorname{Ind}_P^G L_j$ for $j < i$. Since there is no more than r such components for each i, the total number of non-isomorphic indecomposable $k(G)$-modules is not greater than $p^s r = |G|$. □

Lemma 4.3. *If P is a non-cyclic p-group, then P contains a normal subgroup N such that $P/N \cong \mathbb{Z}_p \times \mathbb{Z}_p$.*

Proof. If P is abelian, the statement follows from the classification of finite abelian groups. If P is not abelian, then P has a non-trivial center Z, and P/Z is not cyclic. The statement follows by induction on $|P|$. □

Lemma 4.4. *The group $S = \mathbb{Z}_p \times \mathbb{Z}_p$ has an indecomposable representation of dimension n for each $n \in \mathbb{Z}_{\geq 0}$.*

Proof. Let g and h be two generators of S, $\alpha = g - 1$, $\beta = h - 1$. Then $A = k(S) / (\alpha^2, \beta^2, \alpha\beta, \beta\alpha)$ is the subalgebra of $k(Q)$ for Kronecker quiver Q. In particular, one can see easily that every indecomposable representation of Q remains indecomposable after restriction to A. This implies the Lemma. □

Theorem 4.5. *If a p-Sylow subgroup of G is not cyclic, then G has an indecomposable representation of arbitrary high dimension.*

Proof. By Lemma 4.3 and Lemma 4.4, P has an indecomposable representation Y of dimension n for any positive integer n. Decompose $\operatorname{Ind}_P^G Y$ into direct sum of indecomposable $k(G)$-modules. At least one component X contains Y as an indecomposable $k(P)$ component. Hence $\dim X \geq n$. □

Corollary 4.6. *The group algebra $k(G)$ is finitely represented over a field of characteristic p iff a Sylow p-subgroup of G is cyclic.*

FINAL EXAM
MATH 252

Choose and solve one problem from the list below. The exam is due December 14.

Problem 1. Let $\mathrm{PSL}_2(\mathbb{F}_q)$ be the quotient of $\mathrm{SL}_2(\mathbb{F}_q)$ by the center. Compute the table of irreducible characters for the group $\mathrm{PSL}_2(\mathbb{F}_q)$ over $\mathbb{C}$.

Problem 2. Let k be algebraically closed of characteristic 0, G be a finite group, K be a subgroup. The Hecke algebra $H(G,K)$ is the subalgebra of the group algebra $k(G)$

$$H(G,K) = \{u \in k(G)\, | h_1 u h_2 = u \text{ for any } h_1, h_2 \in H\}.$$

(a) Show that $\dim H(G,K)$ equals the number of double cosets $K\backslash G/K$.

(b) Show that $H(G,K)$ is commutative iff the multiplicity of any irreducible representation of G in $\mathrm{Ind}_K^G(\mathrm{triv})$ is not greater than 1.

(c) Prove that $H(G,K)$ is commutative for $G = S_{p+q}$, $K = S_p \times S_q$.

(d) Describe $H(G,K)$ for $G = \mathrm{GL}_3(\mathbb{F}_q)$, K being the subgroup of upper triangular matrices.

Problem 3. Let G be the group of symmetries of an n-dimensional cube in $\mathbb{R}^n$.

(a) Classify irreducible representation of G. Hint: show that G is a semi-direct product of S_n and the normal subgroup H, isomorphic to Z_2^n, and use induction from H.

(b) Let ρ be the permutation representation of G induced by the action of G on the set of vertices of an n-dimensional cube. Decompose ρ into the sum of irreducibles.

Problem 4.

(a) Let R be a ring, S be a subring. Show that if P is a projective S-module, then $\mathrm{Ind}_S^R P = R \otimes_S P$ is a projective R-module.

(b) Let p be a prime number. Describe projective indecomposable and irreducible representations of S_p over $\mathbb{F}_p$. Hint: use induction from S_{p-1}.

(c) For $p = 5$ find dimensions of all irreducible representations.

Problem 5. Show that the Coxeter functor Φ^+ coincides with the functor τ defined in Crawley-Boevey lectures (page 22), i.e. $\Phi^+(X) = \mathrm{DExt}^1(X, k(Q))$ for any X.

Problem 6. Choose your favorite affine quiver different from $\hat{\mathrm{A}}_n$ (if you choose $\widehat{D}_n$ do it for all n) and your favorite orientation. Describe all indecomposable representations, regular simple representations and tubes.

Date: November 26, 2005.

SOLUTIONS OF SELECTED HOMEWORK PROBLEMS
MATH 252

Problem. Let G be the group of matrices

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

where x, y, z are elements of the finite field $\mathbb{F}_5$. Classify irreducible representations of G over $\mathbb{C}$.

Solution. There are 5 conjugacy classes with one element

$$\begin{pmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

for any $y \in \mathbb{F}_5$ and 24 conjugacy classes, each has one representative

$$\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix},$$

for some $x, y \in \mathbb{F}_5$ such that $x \neq 0$ or $y \neq 0$. Let $H = [G, G]$. Then H coincides with the center of G and consists of matrices

$$\begin{pmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Ther is 25 one-dimensional representations, obtained from the representation of $G/H = \mathbb{Z}_5 \times \mathbb{Z}_5$. The remaining four representations have dimension 5, and can be obtained by induction from the subgroup K of matrices

$$M_{x,y} = \begin{pmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $u \in \mathbb{F}_5^*$ and $\chi_u(M_{x,y}) = e^{2\pi uyi/5}$. Then $\rho_u = \mathrm{Ind}_K^G \chi_u$ has dimension 5, $\rho_u \ncong \rho_v$ if $v \neq u$, since the action of the center is different. Finally

$$\langle \chi_{\rho_u}, \chi_{\rho_u} \rangle_G = \langle \mathrm{Res}_K \chi_{\rho_u}, \chi_u \rangle_K = \sum_{t \in \mathbb{F}_5} \langle \chi_u^t, \chi_u \rangle_K = 1,$$

Date: November 29, 2005.

where $\chi_u^t(M_{x,y}) = \chi_u(M_{x,y+xt})$. That proves irreducibility of each ρ_u. Alternatively, one can prove that ρ_u is irreducible by assuming the contrary. Then a subrepresentation must have dimension 1 (divides the order of the group), but this is impossible since $\rho_u(H) \neq 1$.

Problem. Let G be a finite group, r be the number of conjugacy classes in G and s be the number of conjugacy classes in G preserved by the involution $g \to g^{-1}$. Prove that the number of irreducible representations of G over $\mathbb{R}$ is equal to $\frac{r+s}{2}$.

Solution. Let χ be an irreducible character of G over $\mathbb{C}$. If $\chi(g) \in \mathbb{R}$ for all $g \in G$, then there is one irreducible representation of G over $\mathbb{R}$ with character χ (real) or 2χ (quaternionic). If $\chi(g) \notin \mathbb{R}$ at least for one g, then the pair χ and $\bar{\chi}$ produce one irreducible representation of G over $\mathbb{R}$ (complex) with character $\chi + \bar{\chi}$. Hence if m is the number of irreducible representations of G over $\mathbb{R}$ and p is the number of irreducible characters φ such that $\varphi(g) \in \mathbb{R}$ for all g, then $m = p + \frac{r-p}{2} = \frac{r+p}{2}$. Define the linear operator T on the space of class functions by the formula $T\varphi(g) = \varphi(g) + \varphi(g^{-1})$. Then $\operatorname{rk} T = s + \frac{r-s}{2} = \frac{r+s}{2}$. On the other hand, if φ is an irreducible character, then $\bar{\varphi}(g) = \varphi(g^{-1})$, hence $T(\varphi) = \varphi + \bar{\varphi}$. Since irreducible characters form a basis in the space of class function, one obtains $\operatorname{rk} T = m$.

Problem. Let R be the algebra of polynomial differential operators. In other words R is generated by x and $\frac{\partial}{\partial x}$ with relation

$$\frac{\partial}{\partial x}x - x\frac{\partial}{\partial x} = 1.$$

(The algebra R is called the Weyl algebra.) Let $M = \mathbb{C}[x]$ have a structure of R-module in the natural way. Show that $\operatorname{End}_R(M) = \mathbb{C}$, M is an irreducible R-module and the natural map $R \to \operatorname{End}_{\mathbb{C}}(M)$ is not surjective.

Solution. Note that 1 generates M and if $f \in \operatorname{End}_R(M)$ then $f(p) = pf(1)$ for any $p \in M$. But $\frac{\partial}{\partial x}(f(1)) = 0$. Hence $f(1) = c$ for some $c \in \mathbb{C}$. Therefore $\operatorname{End}_R(M) = \mathbb{C}$. On the other hand, every submodule of M contains 1, therefore M is irreducible. Finally, note that every $d \in R$ has a finite-dimensional kernel. Therefore $\operatorname{End}_{\mathbb{C}}(M) \neq R$.

Problem. Let R be the subalgebra of upper triangular matrices in $\operatorname{Mat}_n(\mathbb{C})$. Classify simple and indecomposable projective modules over R and evaluate $\operatorname{Ext}_R^{\cdot}(M, N)$ for all simple M and N.

Solution. Let E_{ij} denote the elementary matrix with 1 in one place. Then primitive idempotents are E_{ii}, $i = 1, \dots, n$. Indecomposable projectives are $P_i = RE_{ii}$. Note that P_i is isomorphic to the maximal submodule of P_{i+1}. Hence simple modules are $S_i = P_i/P_{i-1}$, if we put $P_0 = 0$. Thus, the complex

$$0 \to P_{i-1} \to P_i \to 0$$

is a projective resolution of S_i. Hence $\operatorname{Ext}^k(S_i, S_j) = 0$ if $k > 1$. Now use that $\operatorname{Hom}_R(P_i, S_i) \cong \mathbb{C}$ and $\operatorname{Hom}_R(P_i, S_j) = 0$ if $i \neq j$, because each P_i has a unique simple quotient isomorphic to S_i. Thus, we obtain $\mathrm{Hom}_R(S_i, S_j) = 0$ if $i \neq j$, $\mathrm{Hom}_R(S_i, S_i) = \mathbb{C}$, $\mathrm{Ext}^1(S_i, S_j) = 0$ if $i \neq j+1$, $\mathrm{Ext}^1(S_{i+1}, S_i) = \mathbb{C}$.

Problem. Let Q be a connected quiver and $k(Q)$ be the path algebra of Q. Show that the center of $k(Q)$ is isomorphic either to k, or to $k[x]$, and that the latter happens only in the case when Q is an oriented cycle.

Solution. Let c be an element of the center of $k(Q)$. Without loss of generality we may assume that c is a linear combination of paths of the same length. Assume that there is an element of the center c of non-zero degree (recall that degree is the length of a path). Write $c = \sum c_{ij}$, where $c_{ij} = e_i c e_j$.

First, we claim that $c_{ij} = 0$ if $i \neq j$. Indeed, if $c_{ij} \neq 0$, then $e_i c = c e_i$ implies $e_i c_{ij} \in k(Q) e_i$, which is impossible.

Next, we claim that if $c_{ii} \neq 0$ for one i, then $c_{jj} \neq 0$ for all j. Indeed, assume the opposite, then, since Q is connected, there exists $\gamma = i \to j$ such that either $c_{ii} = 0$, $c_{jj} \neq 0$ or $c_{jj} = 0$, $c_{ii} \neq 0$. In the former case $e_j \gamma c e_i = \gamma c_{ii} = 0$ and $e_j c \gamma e_i = c_{jj}\gamma \neq 0$, which contradicts $\gamma c = c\gamma$. Similarly, in the latter case $e_j \gamma c e_i \neq 0$, $e_j c \gamma e_i = 0$. Contradiction.

Finally, let γ and $\delta \in Q_1$ and $s(\gamma) = s(\delta) = i$. Then $c\gamma = \gamma c = \gamma c_{ii}$ implies $c_{ii} \in k(Q)\gamma$. By the same reason $c_{ii} \in k(Q)\delta$, which implies $\gamma = \delta$. In the same way, if $t(\gamma) = t(\delta)$, then $\gamma = \delta$. Thus, if there is a central c such that $\deg c > 0$, then Q is one oriented cycle.

Assume first, that Q is not an oriented cycle. The any central element c has degree 0, and therefore $c = \sum b_i e_i$. If $\gamma = i \to j$, then $c\gamma = \gamma c$ implies $b_i = b_j$. But Q is connected, hence $b_1 = \cdots = b_n$. That proves that the center of $k(Q)$ is isomorphic to k.

Let $k(Q)$ be one oriented cycle of length n. Since we already proved that a central element c is a combination of cycles, n divides $\deg c$. A central element of degree sn equals $b \sum sn$-cycles, and hence the center is isomorphic to $k[z]$, where z is the sum of all n-cycles.

Problem. Let $\mathrm{Rep}(a, b, c)$ be the space of all representations of the quiver

$$\bullet \to \bullet \leftarrow \bullet$$

with dimension vector (a, b, c). List all orbits in $\mathrm{Rep}(a, b, c)$. Show that there is only one open orbit. Describe the open orbit O_X in terms of decomposition of X into direct sum of indecomposable representations.

Solution. A point $\mathrm{Rep}(a, b, c)$ is a pair of linear operators $P\colon k^a \to k^b$ and $Q\colon k^c \to k^b$. An orbit is determined by three numbers, $p = \mathrm{rk}\, P$, $q = \mathrm{rk}\, Q$ and $r = \dim(\mathrm{Im}\, P \cap \mathrm{Im}\, Q)$, and we have $p \leq \min(a, b)$, $q \leq \min(b, c)$, $r \leq \min(p, q)$. Positive roots corresponding to indecomposable modules are

$\alpha_1 = (1, 0, 0)$, $\alpha_2 = (0, 1, 0)$, $\alpha_3 = (0, 0, 1)$, $\beta_1 = \alpha_1 + \alpha_2$, $\beta_2 = \alpha_2 + \alpha_3$, $\gamma = \alpha_1 + \alpha_2 + \alpha_3$,

and the decomposition of (P, Q) into the sum of indecomposables is

$$(a - p)\,\alpha_1 + (b - p - q + r)\,\alpha_2 + (c - q)\,\alpha_3 + (p - r)\,\beta_1 + (q - r)\,\beta_2 + r\gamma.$$

By X_ν we denote the indecomposable representation of dimension ν. Check that indecomposable projectives are

$$X_{\beta_1} = Ae_1, X_{\alpha_2} = Ae_2,\ X_{\beta_2} = Ae_3,$$

and the projective resolutions of $X_{\alpha_1}, X_{\alpha_3}$ and X_γ are

$$0 \to X_{\alpha_2} \to X_{\beta_1} \to\ 0,\ 0 \to X_{\alpha_2} \to X_{\beta_2} \to\ 0,$$

$$0 \to X_{\alpha_2} \to X_{\beta_1} \oplus X_{\beta_2} \to\ 0.$$

Therefore,

$$\operatorname{Ext}^1(X_{\alpha_1}, X_{\alpha_2}) = \operatorname{Ext}^1(X_{\alpha_3}, X_{\alpha_2}) = \operatorname{Ext}^1(X_{\alpha_1}, X_{\beta_2}) = \operatorname{Ext}^1(X_{\alpha_3}, X_{\beta_1}) = \operatorname{Ext}^1(X_\gamma, X_{\alpha_2}) = k,$$

all other Ext^1 are trivial.

To determine the open orbit we find possible triples of positive roots without mutual extensions.

$$\{\alpha_1, \beta_1, \gamma\}, \{\alpha_3, \beta_2, \gamma\}, \{\alpha_1, \alpha_3, \gamma\}, \{\beta_1, \beta_2, \gamma\}, \{\beta_1, \beta_2, \alpha_2\}.$$

The open orbit in $\operatorname{Rep}(a, b, c)$ is a combination of one of these triples, here $x = (a, b, c)$:

1. If $a \geq b \geq c$, then $x = (a-b)\alpha_1 + (b-c)\beta_1 + c\gamma$;
2. If $a \leq b \leq c$, then $x = (c-b)\alpha_3 + (b-a)\beta_1 + a\gamma$;
3. If $a, c \geq b$, then $x = (a-b)\alpha_1 + (c-b)\alpha_3 + b\gamma$;
4. If $a, c \leq b, a + c \geq b$, then $x = (b-c)\beta_1 + (b-a)\beta_2 + (a+c-b)\gamma$;
5. If $a, c, a + c \leq b$, then $x = a\beta_1 + c\beta_2 + (b-a-c)\alpha_2$.